# Computational Fluid Dynamics (CFD)

## *Introductory Course*

### Session 02 – Review of Vector Calculus

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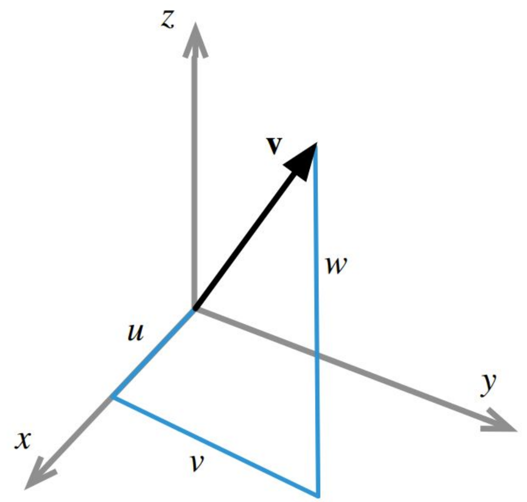
# Syllabus :

- Session 01 – Basic Concepts of CFD
- **Session 02 – Review of Vector Calculus**
- Session 03 – Introduction to Numerical Methods
- Session 04 – Mathematical Description of Physical Phenomena – Part 01
- Session 05 – Mathematical Description of Physical Phenomena – Part 02

# 1. Vector and Vector Operations

- Velocity Vector as the frequently used
  - Unit vectors
  - Column format
  - Transpose of vectors
  - The Magnitude of the vector

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$




$$\mathbf{v} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\mathbf{v}^{\mathrm{T}} = \begin{bmatrix} u & v & w \end{bmatrix}$$

$$\|\mathbf{v}\| = \sqrt{u^2 + v^2 + w^2}$$

## 1. Vector and Vector Operations

- Sum of two vectors

$$\left.\begin{aligned}\mathbf{v}_1 &= u_1\mathbf{i} + v_1\mathbf{j} + w_1\mathbf{k}\\ \mathbf{v}_2 &= u_2\mathbf{i} + v_2\mathbf{j} + w_2\mathbf{k}\end{aligned}\right\} \Rightarrow \mathbf{v}_1 + \mathbf{v}_2 = (u_1 + u_2)\mathbf{i} + (v_1 + v_2)\mathbf{j} + (w_1 + w_2)\mathbf{k}$$

$$\mathbf{v}_1 = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} \Rightarrow \mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} u_1 + u_2 \\ v_1 + v_2 \\ w_1 + w_2 \end{bmatrix}$$

- Multiplication of a vector by a scalar

$$\begin{aligned} s\mathbf{v} &= s(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \\ &= su\mathbf{i} + sv\mathbf{j} + sw\mathbf{k} = \begin{bmatrix} su \\ sv \\ sw \end{bmatrix} \end{aligned}$$

# 1. Vector and Vector Operations

- The Dot Product of Two Vectors

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\| \|\mathbf{v}_2\| \cos(\mathbf{v}_1, \mathbf{v}_2)$$

  - Definition
  - The product is scalar
  - The basic definitions

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$$
$$\mathbf{i}\cdot\mathbf{j} = \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = \mathbf{k}\cdot\mathbf{j} = 0$$

  - Other Definition

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (u_1\mathbf{i} + v_1\mathbf{j} + w_1\mathbf{k}) \cdot (u_2\mathbf{i} + v_2\mathbf{j} + w_2\mathbf{k}) = u_1u_2 + v_1v_2 + w_1w_2$$

  - Vector Magnitude using dot product

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{u^2 + v^2 + w^2}$$

# 1. Vector and Vector Operations

- The Unit Direction Vector

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = \|\mathbf{v}_1\|\|\mathbf{v}_2\|\cos(\mathbf{v}_1, \mathbf{v}_2)$$

  - A unit vector ev in the direction of v can be derived from the definition of the dot product as

$$\left.\begin{aligned} \mathbf{v}\cdot\mathbf{v} &= \|\mathbf{v}\|\|\mathbf{v}\|\overbrace{\cos(\mathbf{v},\mathbf{v})}^{=1} = \|\mathbf{v}\|^2 \Rightarrow \mathbf{v}\cdot\frac{\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{v}\| \\ \mathbf{v}\cdot\mathbf{e}_{\mathrm{v}} &= \|\mathbf{v}\|\underbrace{\|\mathbf{e}_{\mathrm{v}}\|}_{=1}\underbrace{\cos(\mathbf{v},\mathbf{e}_{\mathrm{v}})}_{=1} = \|\mathbf{v}\| \Rightarrow \mathbf{v}\cdot\mathbf{e}_{\mathrm{v}} = \|\mathbf{v}\| \end{aligned}\right\} \Rightarrow \mathbf{e}_{\mathrm{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \qquad (2.11)$$

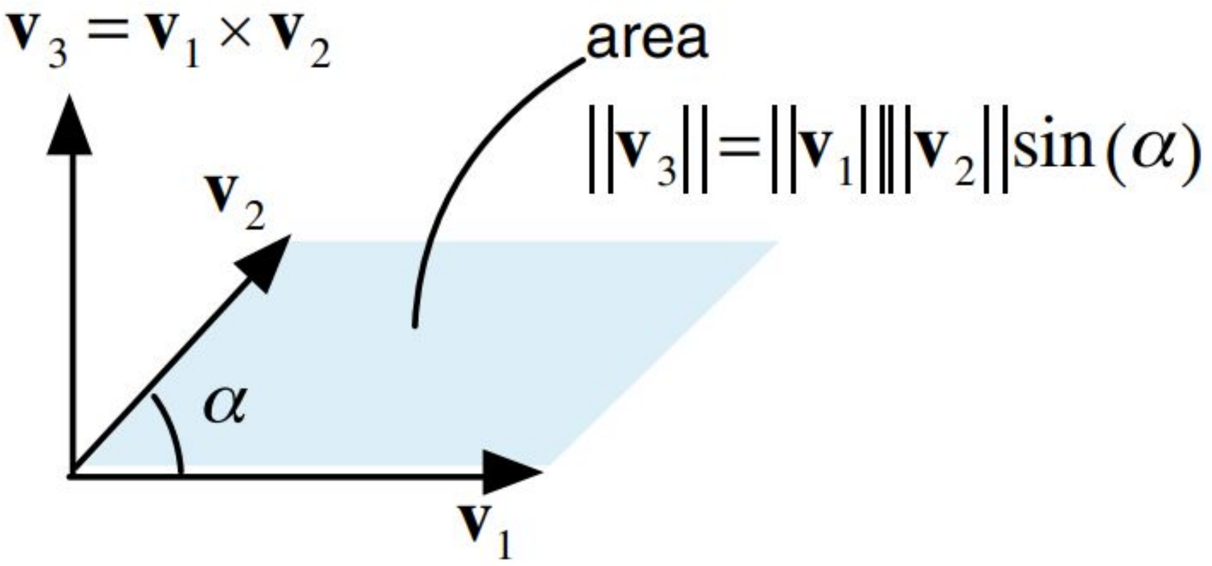
## 1. Vector and Vector Operations

- The Cross Product of Two Vectors

$$\|\mathbf{v}_3\| = \|\mathbf{v}_1 \times \mathbf{v}_2\| = \|\mathbf{v}_1\|\|\mathbf{v}_2\||\sin(\mathbf{v}_1, \mathbf{v}_2)|,$$

- Basic Characteristics:

$$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0 \qquad \mathbf{i}\times\mathbf{j} = \mathbf{k} = -\mathbf{j}\times\mathbf{i}$$
$$\mathbf{j}\times\mathbf{k} = \mathbf{i} = -\mathbf{k}\times\mathbf{j} \qquad \mathbf{k}\times\mathbf{i} = \mathbf{j} = -\mathbf{i}\times\mathbf{k}$$



## 1. Vector and Vector Operations

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0 \qquad \mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}$$
$$\mathbf{j}\times\mathbf{k}=\mathbf{i}=-\mathbf{k}\times\mathbf{j} \qquad \mathbf{k}\times\mathbf{i}=\mathbf{j}=-\mathbf{i}\times\mathbf{k}$$

- The Cross Product of Two Vectors

$$\begin{aligned}\mathbf{v}_1\times\mathbf{v}_2 &= (u_1\mathbf{i}+v_1\mathbf{j}+w_1\mathbf{k})\times(u_2\mathbf{i}+v_2\mathbf{j}+w_2\mathbf{k})\\ &= u_1u_2\mathbf{i}\times\mathbf{i}+u_1v_2\mathbf{i}\times\mathbf{j}+u_1w_2\mathbf{i}\times\mathbf{k}\\ &\quad +v_1u_2\mathbf{j}\times\mathbf{i}+v_1v_2\mathbf{j}\times\mathbf{j}+v_1w_2\mathbf{j}\times\mathbf{k}\\ &\quad +w_1u_2\mathbf{k}\times\mathbf{i}+w_1v_2\mathbf{k}\times\mathbf{j}+w_1w_2\mathbf{k}\times\mathbf{k}\\ &= u_1u_2 0+u_1v_2\mathbf{k}+u_1w_2(-\mathbf{j})\\ &\quad +v_1u_2(-\mathbf{k})+v_1v_2 0+v_1w_2\mathbf{i}\\ &\quad +w_1u_2\mathbf{j}+w_1v_2(-\mathbf{i})+w_1w_2 0\\ &= (v_1w_2-v_2w_1)\mathbf{i}-(u_1w_2-u_2w_1)\mathbf{j}+(u_1v_2-u_2v_1)\mathbf{k}\end{aligned}$$

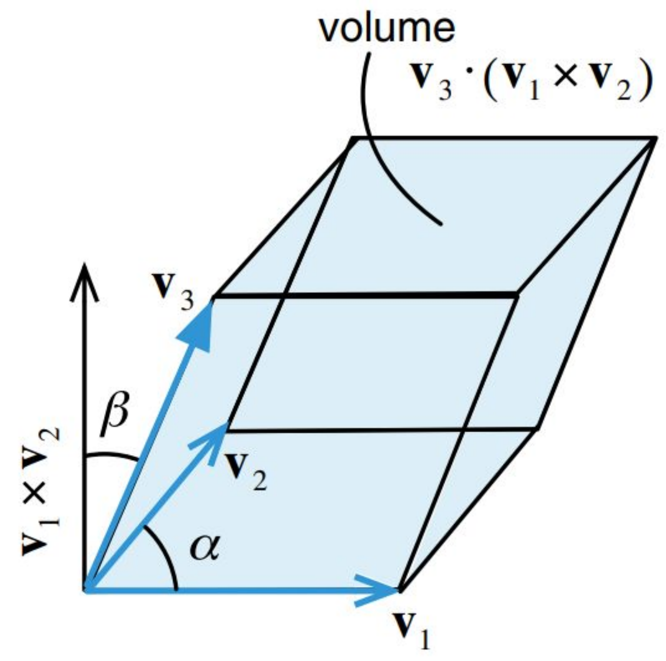
- Or using Determinant

$$\mathbf{v}_1\times\mathbf{v}_2=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ u_1 & v_1 & w_1\\ u_2 & v_2 & w_2\end{vmatrix}=\begin{bmatrix}v_1w_2-v_2w_1\\ u_2w_1-u_1w_2\\ u_1v_2-u_2v_1\end{bmatrix}$$

# 1. Vector and Vector Operations

-  Scalar Triple Product
-  Geometric Interpretation

$$(\mathbf{v}_1 \cdot [\mathbf{v}_2 \times \mathbf{v}_3]) = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}$$



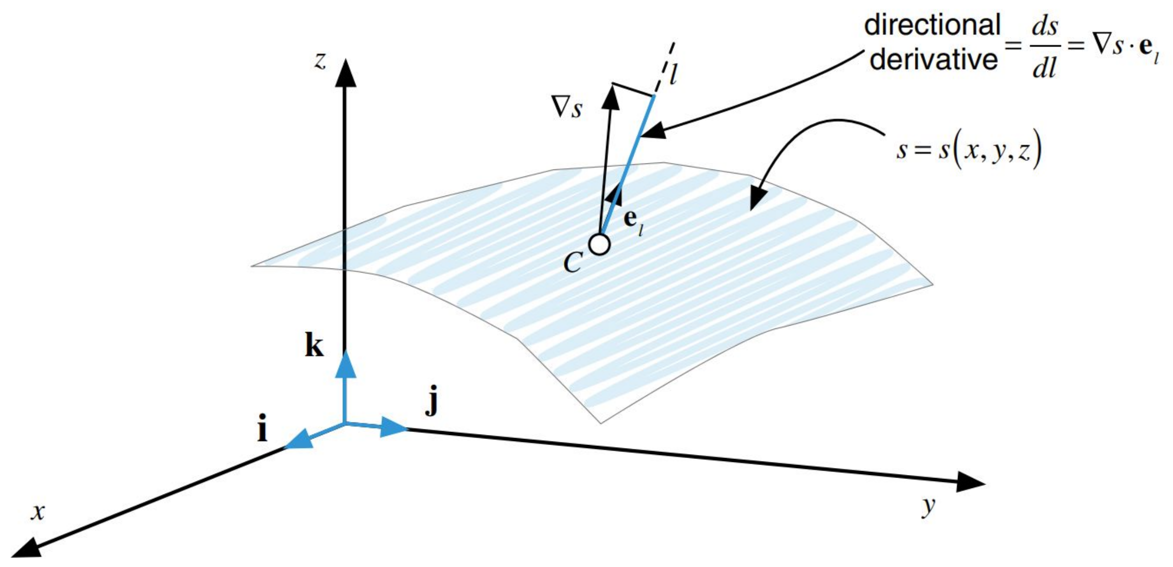
## 1. Vector and Vector Operations

- Gradient of a Scalar and Directional Derivatives
  - “nabla” Operator
  - Gradient of scalar s
  - Projection of $\nabla s$ in a certain direction

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

$$\nabla s = \frac{\partial s}{\partial x}\mathbf{i} + \frac{\partial s}{\partial y}\mathbf{j} + \frac{\partial s}{\partial z}\mathbf{k}$$

$$\frac{ds}{dl} = \nabla s \cdot \mathbf{e}_l = \|\nabla s\| \cos(\nabla s, \mathbf{e}_l)$$



# 1. Vector and Vector Operations

- Example 01

Let $f(x, y, z) = x^2y + y^2z + z^2x$

(a) *find* $\nabla f$ *at point* $(3,2,0)$.

(b) *find the derivative at point* $(3,2,0)$ *along the direction* $(1,2,2)$.

# 1. Vector and Vector Operations

- Solution of Example 01

(a) $\dfrac{\partial f}{\partial x} = 2xy + z^2 \quad \dfrac{\partial f}{\partial y} = x^2 + 2yz \quad \dfrac{\partial f}{\partial z} = y^2 + 2xz$

$$\nabla f = \left(2xy + z^2\right)\mathbf{i} + \left(x^2 + 2yz\right)\mathbf{j} + \left(y^2 + 2xz\right)\mathbf{k}$$

Thus

$$\nabla f\big|_{(3,2,0)} = 12\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$$

# 1. Vector and Vector Operations

- Solution of Example 01

(b) The unit vector along direction $(1, 2, 2)$ is

$$\mathbf{e}_l = \frac{1\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3}$$

The derivative along the direction $(1, 2, 2)$ is

$$\begin{aligned}\left.\frac{df}{dl}\right|_{(3,2,0)} &= \nabla f|_{(3,2,0)} \cdot \mathbf{e}_l \\ &= (12\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}) \cdot \frac{1\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}}{3} \\ &= (12 + 18 + 8)/3 = 38/3\end{aligned}$$

## 1. Vector and Vector Operations

- Operations on the Nabla Operator
  - Divergence of a Vector
  - Laplacian of scalars and vectors
  - Curl of a vector

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\nabla \cdot (\nabla s) = \nabla^2 s = \frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + \frac{\partial^2 s}{\partial z^2}$$

$$\nabla^2 \mathbf{v} = (\nabla^2 u)\mathbf{i} + (\nabla^2 v)\mathbf{j} + (\nabla^2 w)\mathbf{k}$$

$$\begin{aligned}\nabla \times \mathbf{v} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{k}\end{aligned}$$

## 1. Vector and Vector Operations

- Example 02

*Find the divergence of* $\mathbf{v} = (u, v, w) = (3x, 2xy, 4z)$

divergence of $\mathbf{v}$ is obtained as

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
$$= 3 + 2x + 4$$
$$= 7 + 2x$$

## 2. Matrices and Matrix Operations

- Definition.
- Reduction to Vectors and Scalars
- Transpose of a Matrix
- Addition and Subtraction
- Multiplication by a Scalar
- Matrix Multiplication

$$\begin{array}{cc} i & j\rightarrow \quad 1 \quad 2 \quad 3 \\ \downarrow & \\ \begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix} -1 & 2 & -4 \\ 5 & 4 & 7 \\ 0 & 12 & -2 \\ 3 & 6 & 3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} = \left[a_{ij}\right] \end{array}$$

$$\mathbf{A} = [a_{ij}] \Rightarrow s\mathbf{A} = [sa_{ij}]$$

$$p_{ij} = \sum_{k=1}^{X} a_{ik} b_{kj}$$

$$\mathbf{A} = [a_{ij}] \Rightarrow \mathbf{A}^{\mathrm{T}} = [a_{ji}]$$

## 2. Matrices and Matrix Operations

- Square Matrix
  - Main Diagonal
  - Symmetric Matrix
- Special Matrices
  - Identity Matrix
  - Upper and Lower Triangular Matrices

$$a_{ij} = -a_{ji}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\mathbf{U} = \begin{cases} u_{ij} & i \le j \\ 0 & i > j \end{cases}$$

# 2. Matrices and Matrix Operations

- Using Matrices to Describe Systems of Equations

$$
\begin{aligned}
a_{11}\phi_1 + a_{12}\phi_2 + a_{13}\phi_3 + \ldots + a_{1N}\phi_N &= b_1 \\
a_{21}\phi_1 + a_{22}\phi_2 + a_{23}\phi_3 + \ldots + a_{2N}\phi_N &= b_2 \\
a_{31}\phi_1 + a_{32}\phi_2 + a_{33}\phi_3 + \ldots + a_{3N}\phi_N &= b_3 \\
\vdots \quad\quad \vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad\quad & \vdots \\
a_{N1}\phi_1 + a_{N2}\phi_2 + a_{N3}\phi_3 + \ldots + a_{NN}\phi_N &= b_N
\end{aligned}
$$

$$\mathbf{A\phi} = \mathbf{b}$$

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1N} \\
a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2N} \\
a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3N} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{N1} & a_{N2} & a_{N3} & \cdots & \cdots & a_{NN}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \\ \vdots \\ \phi_N
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3 \\ \vdots \\ \vdots \\ b_N
\end{bmatrix}
$$

## 2. Matrices and Matrix Operations

- The Determinant of a Square Matrix
  - Order 2

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(\mathbf{A}) = a_{11}a_{22} - a_{21}a_{12}$$

## 2. Matrices and Matrix Operations

- Eigenvectors and Eigenvalues

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \Rightarrow \mathbf{A}\mathbf{v} = \lambda\mathbf{I}\mathbf{v} \Rightarrow (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$$

$$\mathbf{A} - \lambda\mathbf{I} = 0 \Rightarrow \det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\det\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & \cdots & a_{1N} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & \cdots & a_{2N} \\ a_{31} & a_{32} & a_{33}-\lambda & \cdots & \cdots & a_{3N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & \cdots & a_{NN}-\lambda \end{bmatrix} = 0$$

## 2. Tensors

- Stress Tensor

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix}$$

$$\boldsymbol{\tau} = \mathbf{ii}\tau_{xx} + \mathbf{ij}\tau_{xy} + \mathbf{ik}\tau_{xz} + \mathbf{ji}\tau_{yx} + \mathbf{jj}\tau_{yy} + \mathbf{jk}\tau_{yz} + \mathbf{ki}\tau_{zx} + \mathbf{kj}\tau_{zy} + \mathbf{kk}\tau_{zz}$$

- Dyadic product of a vector by itself

$$\left.\begin{aligned} \{\mathbf{vv}\} &= (u\mathbf{i} + v\mathbf{j} + w\mathbf{k})(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \\ &= \mathbf{ii}uu + \mathbf{ij}uv + \mathbf{ik}uw + \\ &\quad \mathbf{ji}vu + \mathbf{jj}vv + \mathbf{jk}vw + \\ &\quad \mathbf{ki}wu + \mathbf{kj}wv + \mathbf{kk}ww \end{aligned}\right\} \Rightarrow \{\mathbf{vv}\} = \begin{bmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{bmatrix}$$

## 2. Tensors

- Stress Tensor
  - Gradient of a vector

$$
\left.
\begin{aligned}
\{\nabla \mathbf{v}\} &= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) \\
&= \mathbf{ii}\frac{\partial u}{\partial x} + \mathbf{ij}\frac{\partial v}{\partial x} + \mathbf{ik}\frac{\partial w}{\partial x} + \\
&\quad \mathbf{ji}\frac{\partial u}{\partial y} + \mathbf{jj}\frac{\partial v}{\partial y} + \mathbf{jk}\frac{\partial w}{\partial y} + \\
&\quad \mathbf{ki}\frac{\partial u}{\partial z} + \mathbf{kj}\frac{\partial v}{\partial z} + \mathbf{kk}\frac{\partial w}{\partial z}
\end{aligned}
\right\} \Rightarrow \{\nabla \mathbf{v}\} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}
$$

## 2. Tensors

- Stress Tensor
  - Dot product of a tensor by a vector

$$[\boldsymbol{\tau}\cdot\mathbf{v}] = \begin{pmatrix} \mathbf{ii}\tau_{xx} + \mathbf{ij}\tau_{xy} + \mathbf{ik}\tau_{xz} + \mathbf{ji}\tau_{yx} + \\ \mathbf{jj}\tau_{yy} + \mathbf{jk}\tau_{yz} + \mathbf{ki}\tau_{zx} + \mathbf{kj}\tau_{zy} + \mathbf{kk}\tau_{zz} \end{pmatrix} \cdot (u\mathbf{i} + v\mathbf{j} + w\mathbf{k})$$

$$[\boldsymbol{\tau}\cdot\mathbf{v}] = (\tau_{xx}u + \tau_{xy}v + \tau_{xz}w)\mathbf{i} + (\tau_{yx}u + \tau_{yy}v + \tau_{yz}w)\mathbf{j} + (\tau_{zx}u + \tau_{zy}v + \tau_{zz}w)\mathbf{k}$$

$$[\boldsymbol{\tau}\cdot\mathbf{v}] = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \tau_{xx}u + \tau_{xy}v + \tau_{xz}w \\ \tau_{yx}u + \tau_{yy}v + \tau_{yz}w \\ \tau_{zx}u + \tau_{zy}v + \tau_{zz}w \end{bmatrix}$$

## 2. Tensors

- Stress Tensor
  - Divergence of a Tensor

$$[\nabla \cdot \boldsymbol{\tau}] = \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\mathbf{i} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\mathbf{j} + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)\mathbf{k}$$

## 2. Tensors

- Stress Tensor
  - Double Dot product of two tensors

$$(\boldsymbol{\tau} : \nabla \mathbf{v}) = \begin{pmatrix} \mathbf{ii}\tau_{xx} + \mathbf{ij}\tau_{xy} + \mathbf{ik}\tau_{xz} + \\ \mathbf{ji}\tau_{yx} + \mathbf{jj}\tau_{yy} + \mathbf{jk}\tau_{yz} + \\ \mathbf{ki}\tau_{zx} + \mathbf{kj}\tau_{zy} + \mathbf{kk}\tau_{zz} \end{pmatrix} : \begin{pmatrix} \mathbf{ii}\frac{\partial u}{\partial x} + \mathbf{ij}\frac{\partial v}{\partial x} + \mathbf{ik}\frac{\partial w}{\partial x} + \\ \mathbf{ji}\frac{\partial u}{\partial y} + \mathbf{jj}\frac{\partial v}{\partial y} + \mathbf{jk}\frac{\partial w}{\partial y} + \\ \mathbf{ki}\frac{\partial u}{\partial z} + \mathbf{kj}\frac{\partial v}{\partial z} + \mathbf{kk}\frac{\partial w}{\partial z} \end{pmatrix}$$

$$(\boldsymbol{\tau} : \nabla \mathbf{v}) = \tau_{xx}\frac{\partial u}{\partial x} + \tau_{xy}\frac{\partial u}{\partial y} + \tau_{xz}\frac{\partial u}{\partial z} + \tau_{yx}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{yz}\frac{\partial v}{\partial z} + \tau_{zx}\frac{\partial w}{\partial x} + \tau_{zy}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial w}{\partial z}$$

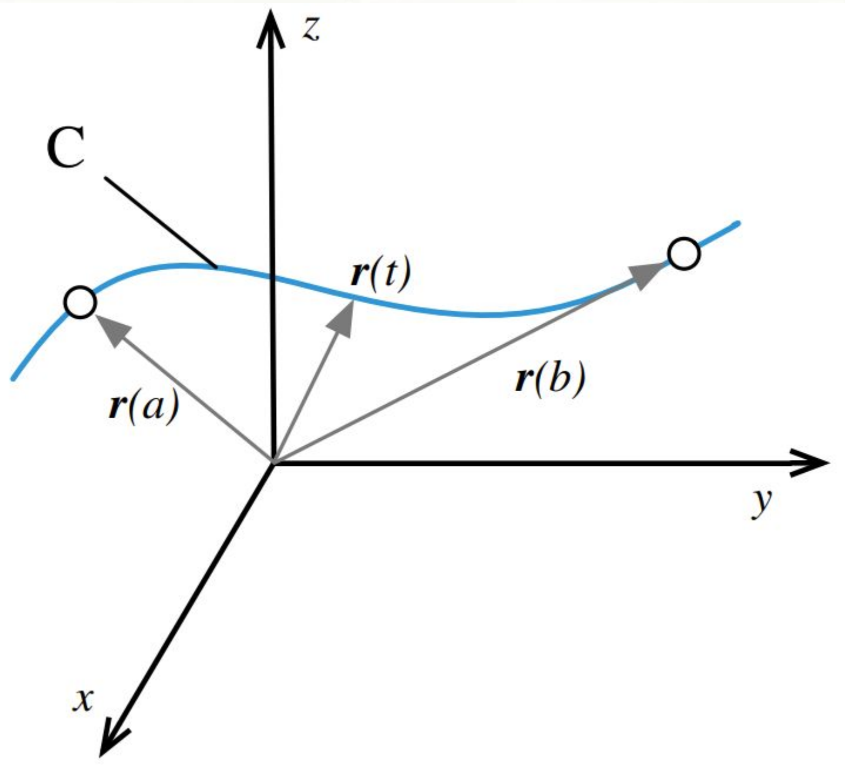
# 3. Fundamental Theorems of Vector Calculus

- Gradient Theorem for Line Integrals

It relates a line integral to the values of a function at its endpoints

$$\mathbf{r}(t) = \mathbf{r}[x(t), y(t), z(t)] \text{ for } a \leq t \leq b$$

$$\int_C \nabla s \cdot d\mathbf{r} = s(r(b)) - s(r(a))$$



## 3. Fundamental Theorems of Vector Calculus

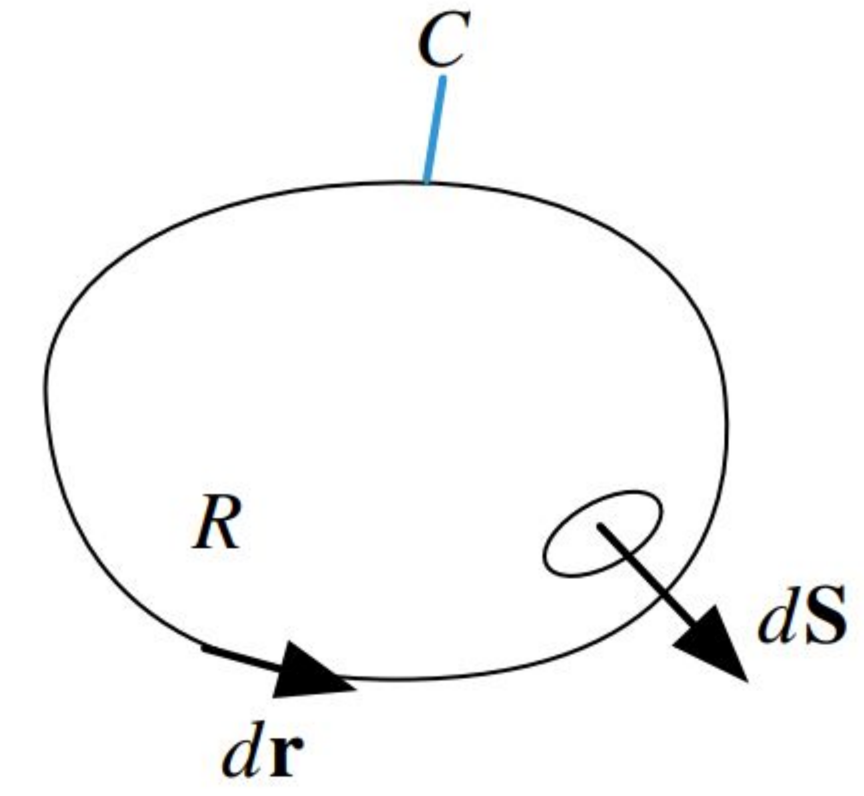
- Green's Theorem

It expresses the contour integral of a simple closed curve C in terms of the double integral of the two dimensional region R bounded by C

$$\oint_C (udx + vdy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$

$$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} \qquad \mathbf{v} = u\mathbf{i} + v\mathbf{j} \qquad d\mathbf{S} = dxdy\mathbf{k}$$

$$\oint_C \mathbf{v} \cdot d\mathbf{r} = \iint_R [\nabla \times \mathbf{v}] \cdot d\mathbf{S}$$

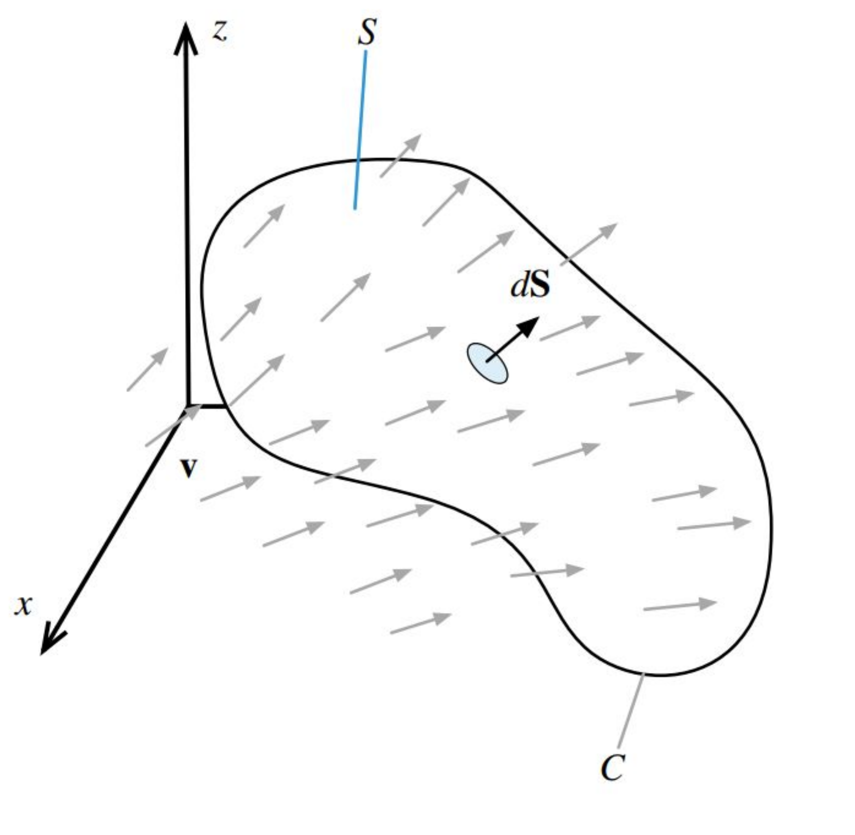


# 3. Fundamental Theorems of Vector Calculus

- Stoke's Theorem

It is a higher dimensional version of Green's Theorem

$$\int_S [\nabla \times \mathbf{v}] \cdot d\mathbf{S} = \oint_C \mathbf{v} \cdot d\mathbf{r}$$

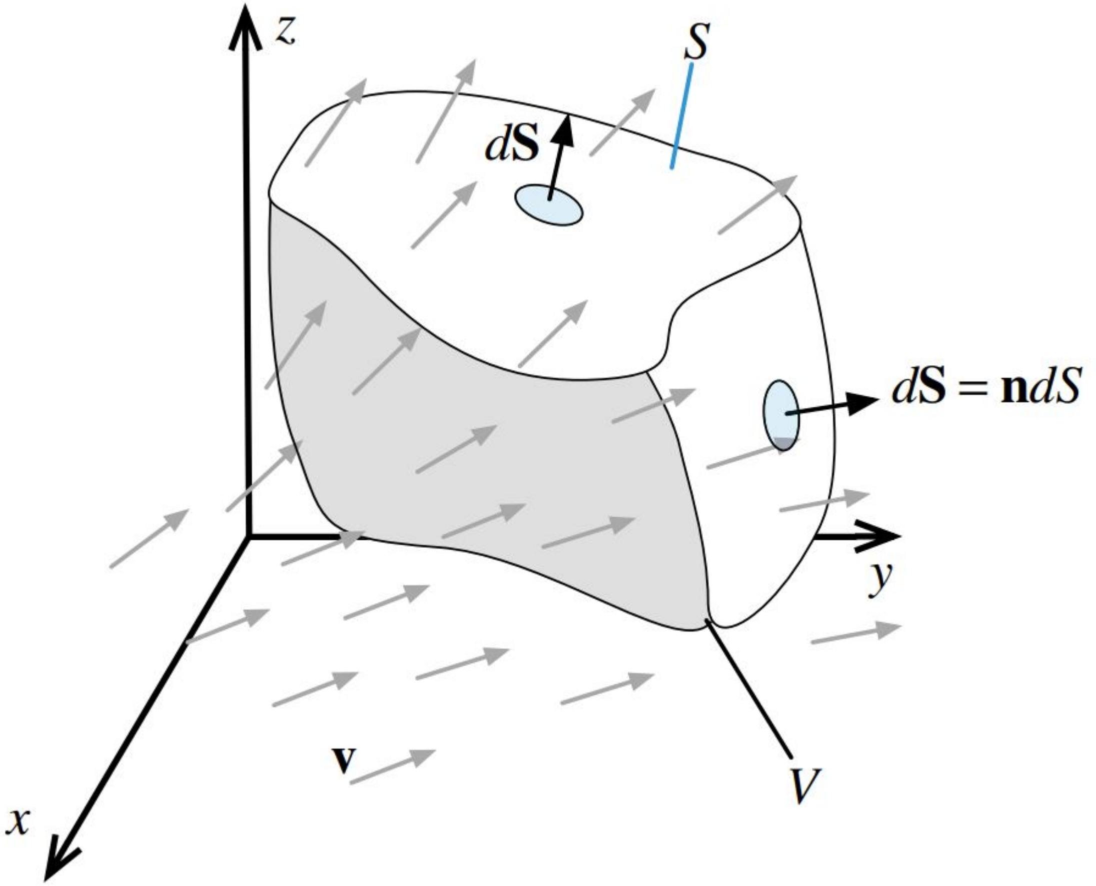


# 3. Fundamental Theorems of Vector Calculus

- Divergence Theorem

  It implies that the net flux of a vector field through a closed surface is equal to the total volume of all sources and sinks (i.e., the volume integral of its divergence) over the region inside the surface.

$$\int_V (\nabla \cdot \mathbf{v}) dV = \oint_S \mathbf{v} \cdot \mathbf{n}\, dS$$

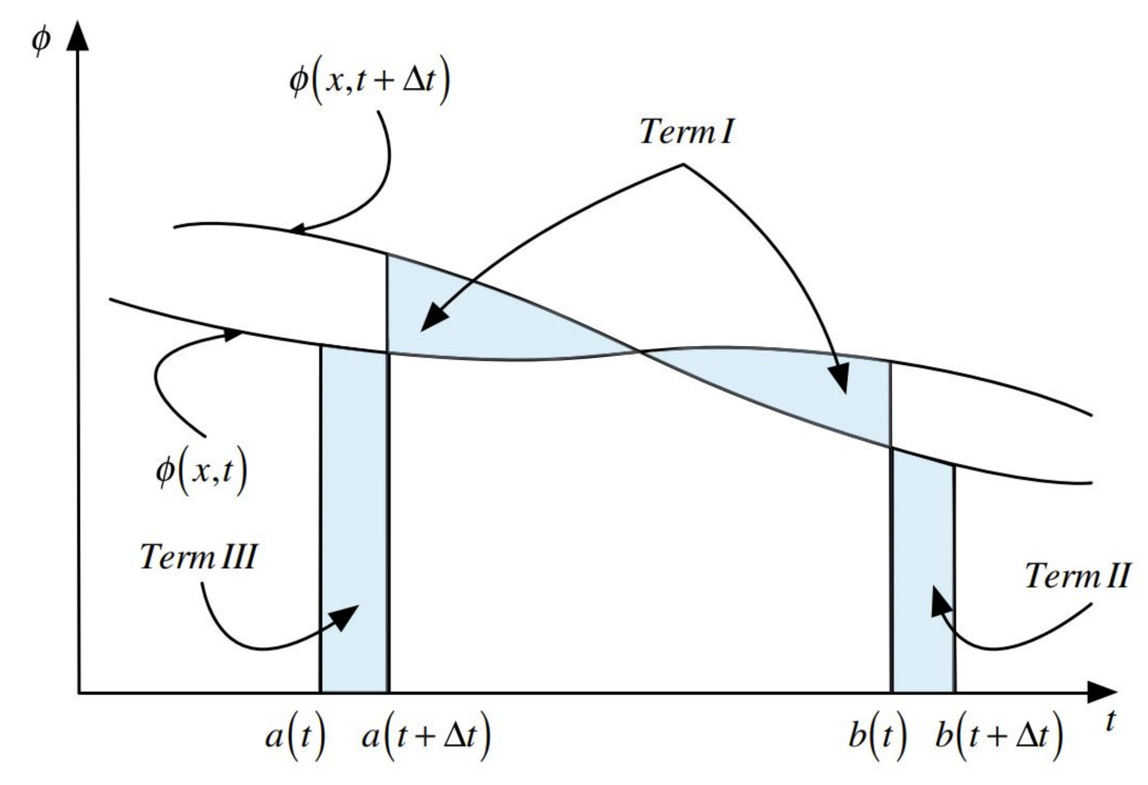


# 3. Fundamental Theorems of Vector Calculus

- Leibniz Integral Theorem

The first term on the right side gives the change in the integral because / is changing with time t, while the second and third terms accounts for the gain and loss in area as the upper and lower bounds are moved, respectively.

$$\frac{d}{dt}\int_{a(t)}^{b(t)}\phi(x,t)dx = \underbrace{\int_{a(t)}^{b(t)}\frac{\partial\phi}{\partial t}dx}_{Term\,I} + \underbrace{\phi(b(t),t)\frac{\partial b}{\partial t}}_{Term\,II} - \underbrace{\phi(a(t),t)\frac{\partial a}{\partial t}}_{Term\,III}$$



Thanks for your
time and attention