Computational Fluid Dynamics (CFD)

Introductory Course

Session 03 – Introduction to Numerical Methods

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Syllabus :

- Session 01 – Basic Concepts of CFD
- Session 02 – Review of Vector Calculus
- **Session 03 – Mathematical Description of Physical Phenomena**

1. Properties of Numerical Solution Methods

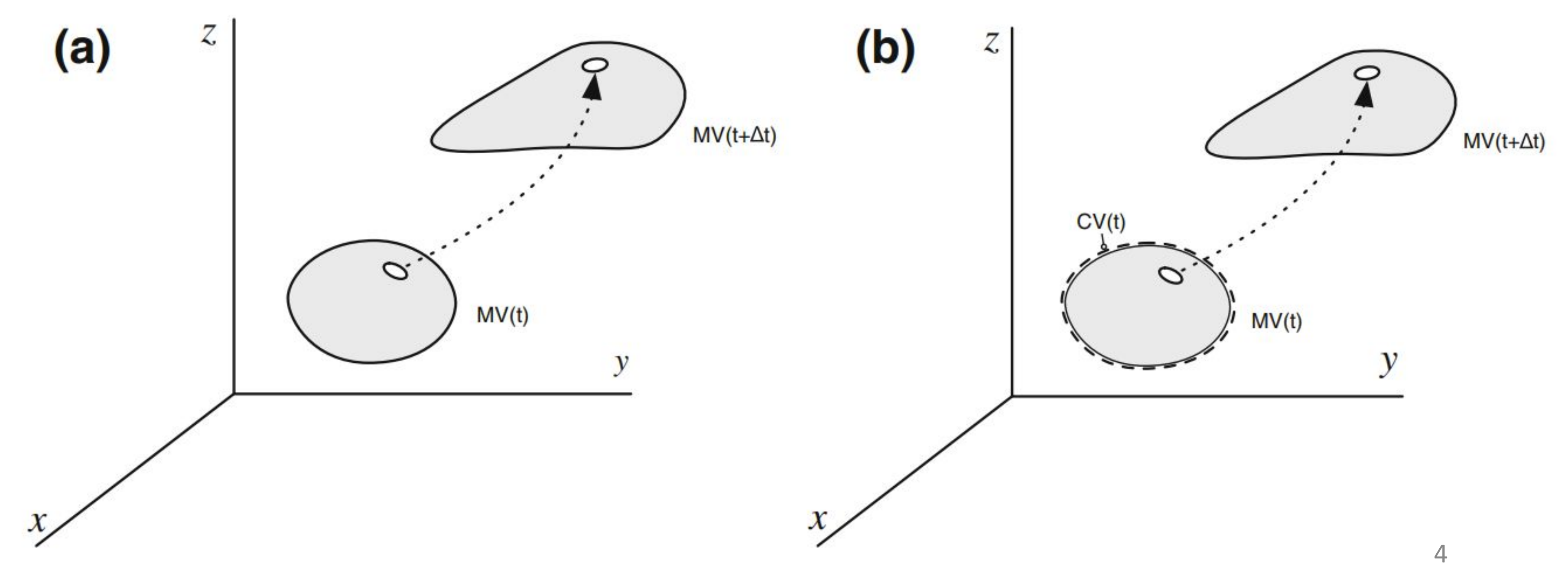
- Consistency
  - Truncation Error

- Stability
  - For Steady Problems
  - For Temporal Problems
  - For Numerical Methods

- Convergence
- Conservation
- Boundedness

2. Eulerian and Lagrangian Description of Conservation Laws

- Lagrangian. Follows the particles of fluid as they move through space and time
- Eulerian. Focuses on specific locations in the flow region as time passes



3. Substantial vs. Local Derivative

- Rate of change of a variable $\phi(t, \mathbf{x}(t))$.
- Eulerian (local) Derivative $(\partial\phi/\partial t)$
- Lagrangian (substantial) $(D\phi/Dt)$

$$\frac{D\phi}{Dt} = \frac{\partial\phi}{\partial t}\frac{dt}{dt} + \frac{\partial\phi}{\partial x}\underbrace{\frac{dx}{dt}}_{u} + \frac{\partial\phi}{\partial y}\underbrace{\frac{dy}{dt}}_{v} + \frac{\partial\phi}{\partial z}\underbrace{\frac{dz}{dt}}_{w}$$

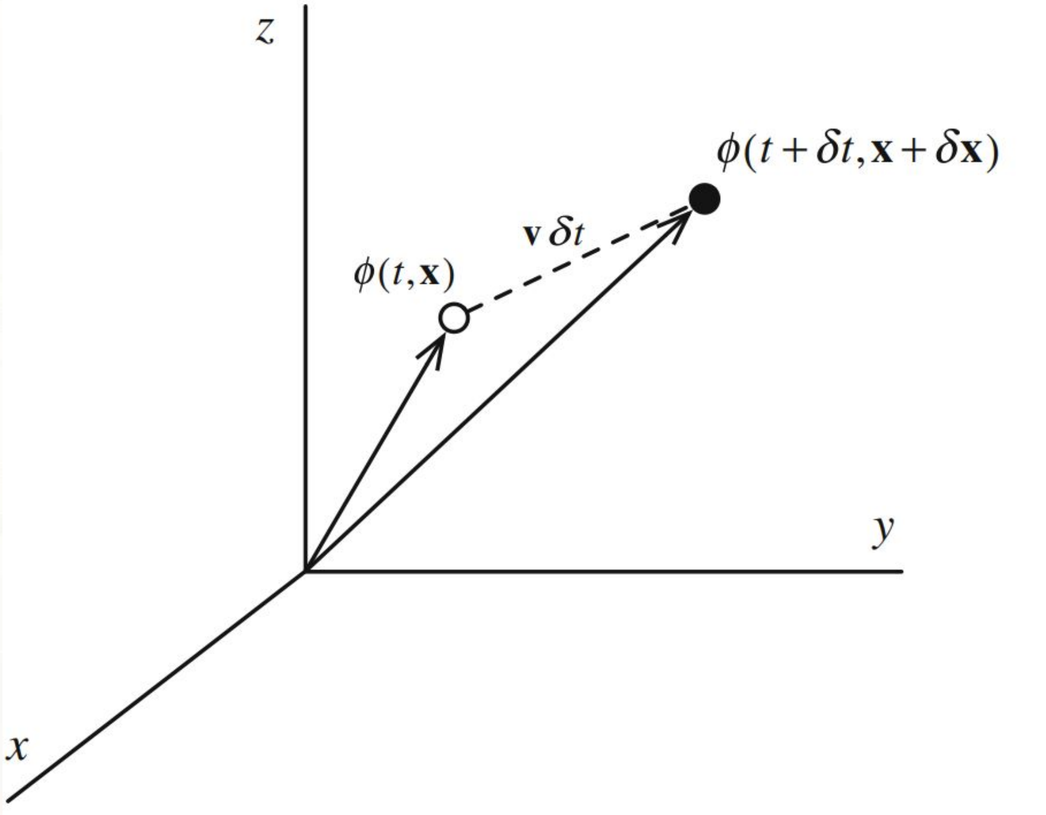
$$= \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z}$$

$$= \underbrace{\frac{\partial\phi}{\partial t}}_{\substack{\text{local rate}\\ \text{of change}}} + \underbrace{\mathbf{v}\cdot\nabla\phi}_{\substack{\text{convective rate}\\ \text{of change}}}$$

3. Substantial vs. Local Derivative

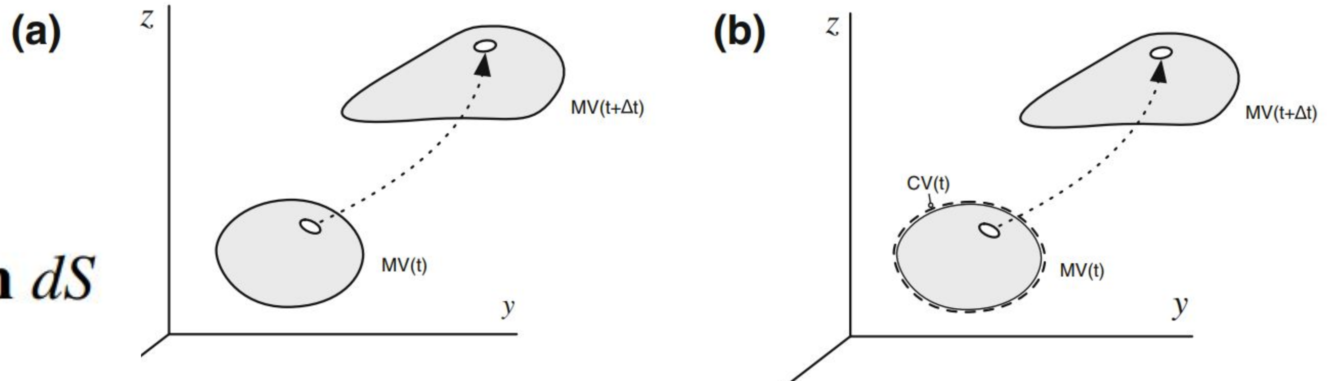
- Rate of change of a variable $\phi(t, \mathbf{x}(t))$.
- Eulerian (local) Derivative $(\partial\phi/\partial t)$
- Lagrangian (substantial) $(D\phi/Dt)$

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}$$



4. Reynolds Transport Theorem

- Is used to express the conservation laws to an Eulerian Approach (Control Volumes)
- Let ***B*** be any property of the fluid (mass, momentum, energy, etc.)
- Thus, intensive value of B will be $b = dB/dm$
- The instantaneous change of B in MV



$$\left(\frac{dB}{dt}\right)_{MV} = \frac{d}{dt}\left(\int_{V(t)} b\rho dV\right) + \int_{S(t)} b\rho \mathbf{v}_r \cdot \mathbf{n}\, dS$$

$$\mathbf{v}_r = \mathbf{v}(t, \mathbf{x}) - \mathbf{v}_s(t, \mathbf{x})$$

4. Reynolds Transport Theorem

- For a fixed CV, $\mathbf{v}_s = 0$, thus using Leibniz rule:

$$\frac{d}{dt}\left(\int_V b\rho \, dV\right) = \int_V \frac{\partial}{\partial t}(b\rho) dV$$

$$\left(\frac{dB}{dt}\right)_{MV} = \int_V \frac{\partial}{\partial t}(b\rho) dV + \int_S b\rho\mathbf{v}\cdot\mathbf{n}\, dS$$

Using Divergence Theorem:

$$\left(\frac{dB}{dt}\right)_{MV} = \int_V \left[\frac{\partial}{\partial t}(\rho b) + \nabla\cdot(\rho\mathbf{v}b)\right] dV$$

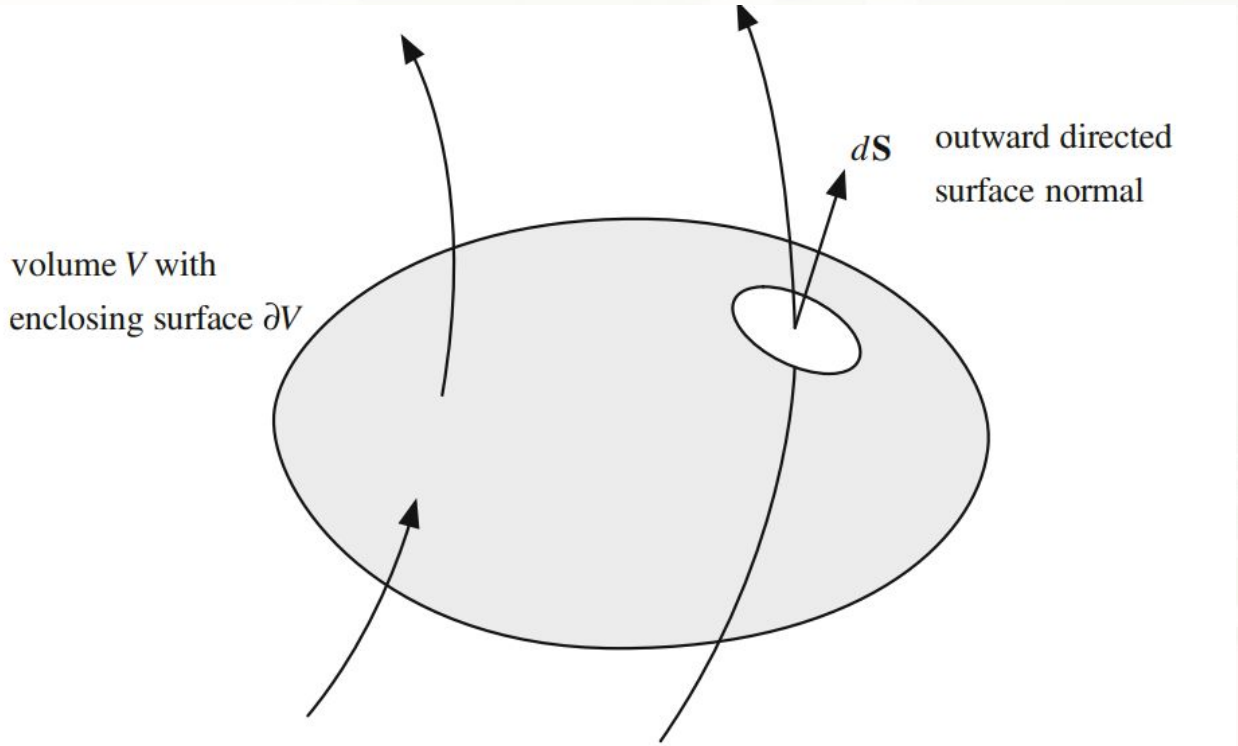
$$\left(\frac{dB}{dt}\right)_{MV} = \int_V \left[\frac{D}{Dt}(\rho b) + \rho b\nabla\cdot\mathbf{v}\right] dV$$

5. Conservation of Mass
Continuity eq.

- What is says

- Using Lagrangian approach:

$$\left(\frac{dm}{dt}\right)_{MV} = 0$$



- Defining :

$$B = m$$
$$b = 1$$

5. Conservation of Mass
Continuity eq.

- Using Reynolds Transport Theorem

$$\int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{v}]\right) dV = 0$$

- For any CV :

$$\frac{\partial \rho}{\partial t} + \nabla \cdot [\rho \mathbf{v}] = 0$$

- Special Cases

$$\nabla \cdot \mathbf{v} = 0$$

6. Conservation of Linear Momentum

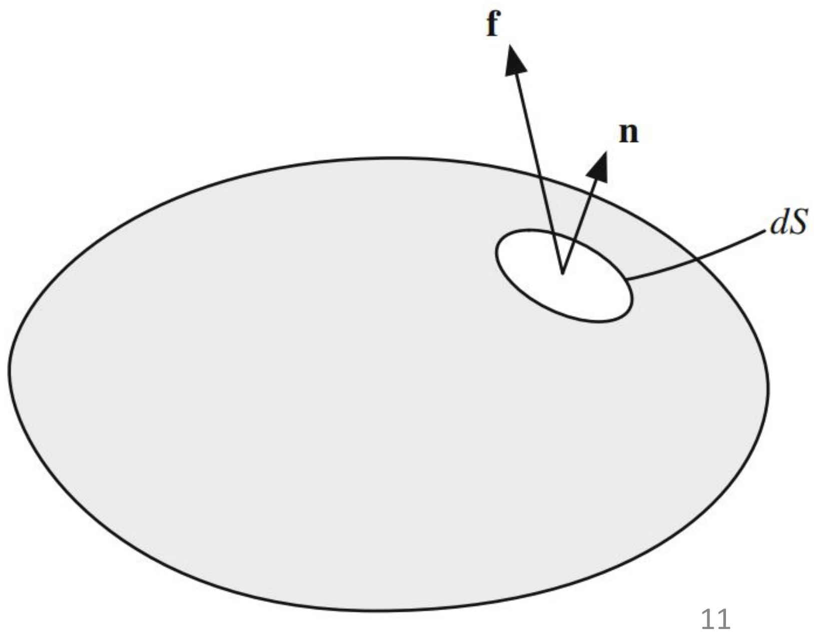
- What it says

- Using Lagrangian approach:

$$\left(\frac{d(m\mathbf{v})}{dt}\right)_{MV} = \left(\int_V \mathbf{f}\, dV\right)_{MV}$$

- For the moving fluid:

$$\left(\int_V \mathbf{f}\, dV\right)_{MV} = \int_V \mathbf{f}\, dV$$



6. Conservation of Linear Momentum

- Using the Reynolds Transport Theorem with $b = \mathbf{v}$,

$$\int_V \left[\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla\cdot\{\rho\mathbf{v}\mathbf{v}\} - \mathbf{f}\right] dV = 0$$

- For any CV :

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla\cdot\{\rho\mathbf{v}\mathbf{v}\} = \mathbf{f}$$

$\rho\mathbf{v}\mathbf{v}$ is the dyadic product,

$$\mathbf{f} = \mathbf{f}_s + \mathbf{f}_b$$

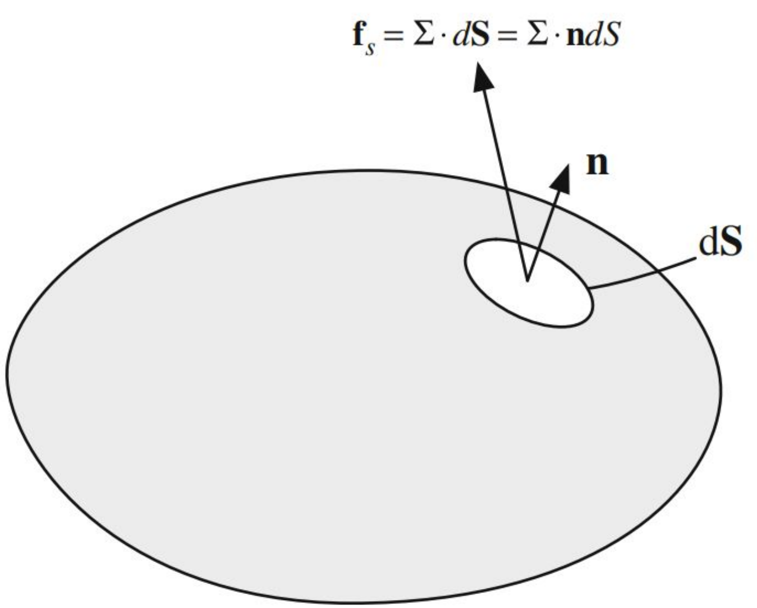
6. Conservation of Linear Momentum
Surface Forces

- The Stress Tensors

$$\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{xz} \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{yz} \\ \Sigma_{zx} & \Sigma_{zy} & \Sigma_{zz} \end{pmatrix}$$

- In Practice:

$$\mathbf{\Sigma} = -\begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix} + \begin{pmatrix} \overbrace{\Sigma_{xx+p}}^{\tau_{xx}} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \overbrace{\Sigma_{yy+p}}^{\tau_{yy}} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \overbrace{\Sigma_{zz+p}}^{\tau_{zz}} \end{pmatrix} = -p\mathbf{I} + \boldsymbol{\tau}$$



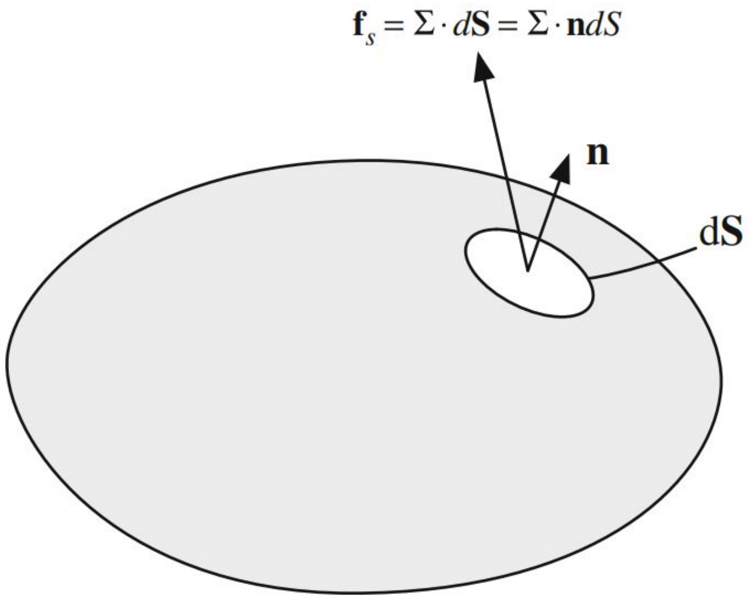
6. Conservation of Linear Momentum
Surface Forces

- Due to the illustration and by applying the divergence theorem:

$$\int_V \mathbf{f}_s \, dV = \int_S \mathbf{\Sigma} \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \mathbf{\Sigma} \, dV$$

- Thus:

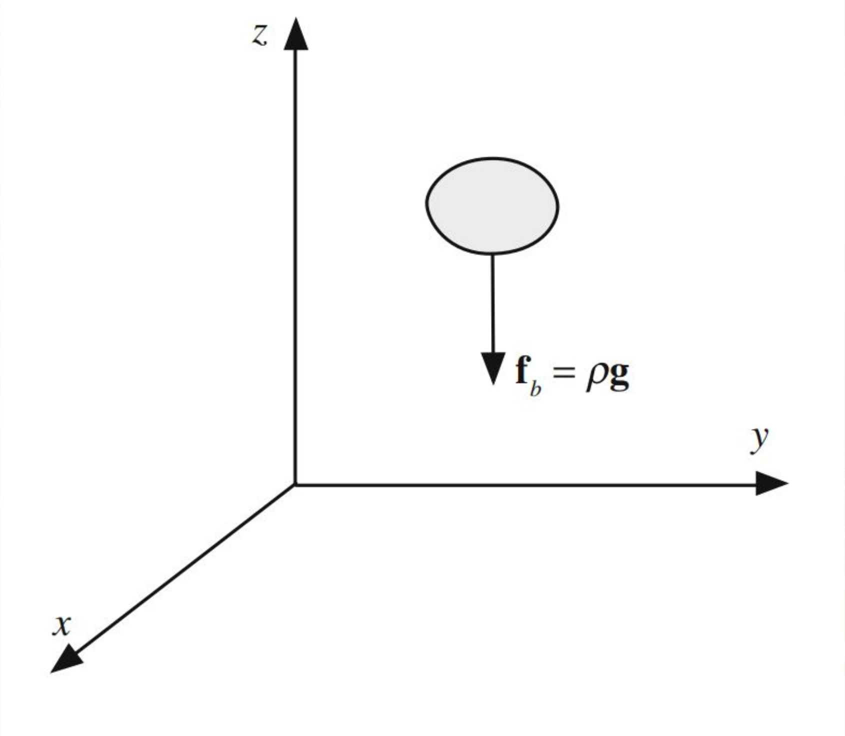
$$\mathbf{f}_s = [\nabla \cdot \mathbf{\Sigma}] = -\nabla p + [\nabla \cdot \boldsymbol{\tau}]$$



6. Conservation of Linear Momentum
Body Forces – Gravitational

- Gravitational Force

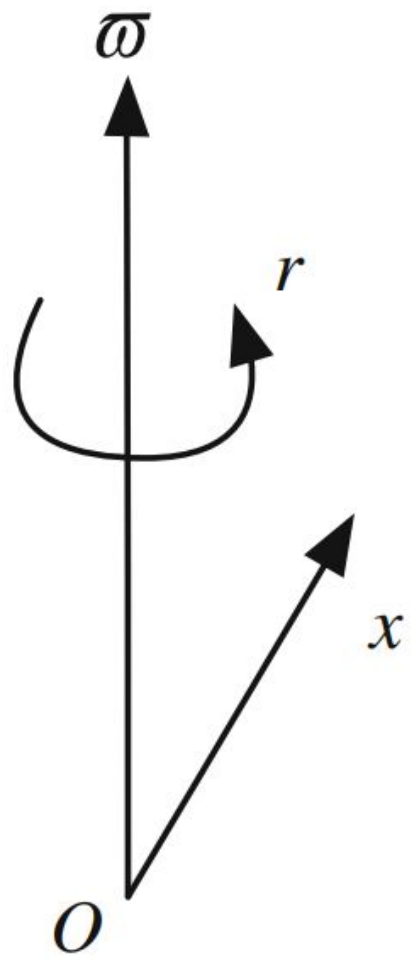
$$\mathbf{f}_b = \rho \mathbf{g}$$



6. Conservation of Linear Momentum
Body Forces – System Rotation

- In a rigid rotating body

$$\mathbf{f}_b = \underbrace{-2\rho[\varpi \times \mathbf{v}]}_{Coriolis\ forces} - \underbrace{\rho[\varpi \times [\varpi \times \mathbf{r}]]}_{Centrifugal\ forces}$$



6. Conservation of Linear Momentum
General form

- Without Considering body forces, electric, and magnetic forces, we’ve come to :

$$\frac{\partial}{\partial t}[\rho \mathbf{v}] + \nabla \cdot \{\rho \mathbf{v}\mathbf{v}\} = -\nabla p + [\nabla \cdot \tau] + \mathbf{f}_b$$

6. Conservation of Linear Momentum
General form – for Newtonian Fluids

- Stress Tensor for a Newtonian Fluid:

$$\boldsymbol{\tau} = \mu\left\{\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right\} + \lambda(\nabla\cdot\mathbf{v})\mathbf{I}$$

  - Where μ is the molecular viscosity coef. And λ is the bulk viscosity coef. And is usually set to $-(2/3)\mu$

  - Thus,

$$\boldsymbol{\tau} = \begin{bmatrix} 2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\mathbf{v} & \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & 2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{v} & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) \\ \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & 2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{v} \end{bmatrix}$$

6. Conservation of Linear Momentum
General form – for Newtonian Fluids

- We need the divergence of the stress tensor

$$[\nabla \cdot \boldsymbol{\tau}] = \nabla \cdot \left[\mu\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}\right)\right] + \nabla(\lambda \nabla \cdot \mathbf{v})$$

$$= \begin{bmatrix} \dfrac{\partial}{\partial x}\left[2\mu\dfrac{\partial u}{\partial x} + \lambda\nabla\cdot\mathbf{v}\right] + \dfrac{\partial}{\partial y}\left[\mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right)\right] + \dfrac{\partial}{\partial z}\left[\mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right)\right] \\ \dfrac{\partial}{\partial x}\left[\mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right)\right] + \dfrac{\partial}{\partial y}\left[2\mu\dfrac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{v}\right] + \dfrac{\partial}{\partial z}\left[\mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right)\right] \\ \dfrac{\partial}{\partial x}\left[\mu\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right)\right] + \dfrac{\partial}{\partial y}\left[\mu\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right)\right] + \dfrac{\partial}{\partial z}\left[2\mu\dfrac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{v}\right] \end{bmatrix}$$

6. Conservation of Linear Momentum
General form – for Newtonian Fluids

- In the closed form

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla\cdot\{\rho\mathbf{v}\mathbf{v}\} = -\nabla p + \nabla\cdot\left\{\mu\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\right]\right\} + \nabla(\lambda\nabla\cdot\mathbf{v}) + \mathbf{f}_b$$

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla\cdot\{\rho\mathbf{v}\mathbf{v}\} = \nabla\cdot\{\mu\nabla\mathbf{v}\} - \nabla p + \underbrace{\nabla\cdot\left\{\mu(\nabla\mathbf{v})^{\mathrm{T}}\right\} + \nabla(\lambda\nabla\cdot\mathbf{v}) + \mathbf{f}_b}_{\mathbf{Q}^{\mathrm{v}}}$$

$$\frac{\partial}{\partial t}[\rho\mathbf{v}] + \nabla\cdot\{\rho\mathbf{v}\mathbf{v}\} = \nabla\cdot\{\mu\nabla\mathbf{v}\} - \nabla p + \mathbf{Q}^{\mathrm{v}}$$

6. Conservation of Linear Momentum
General form – for Newtonian Fluids

- For incompressible flows: $\nabla \cdot \mathbf{v} = 0,$

$$\frac{\partial}{\partial t}[\rho \mathbf{v}] + \nabla \cdot \{\rho \mathbf{v}\mathbf{v}\} = -\nabla p + \nabla \cdot \left\{\mu\left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}}\right]\right\} + \mathbf{f}_b$$

- Divergence of the stress tensor for the incompressible flows:

$$\mu\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x}\right] + \mu\frac{\partial}{\partial y}\left[\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] + \mu\frac{\partial}{\partial z}\left[\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right]$$
$$= \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial yx} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial zx}\right]$$
$$= \mu\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial yx} + \frac{\partial^2 w}{\partial zx}\right]$$
$$= \mu\left[\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$$

6. Conservation of Linear Momentum
General form – for Newtonian Fluids

- And Finally:

$$\frac{\partial}{\partial t}[\rho \mathbf{v}] + \nabla \cdot \{\rho \mathbf{v}\mathbf{v}\} = -\nabla p + \mu \nabla^2 \mathbf{v} + \mathbf{f}_b$$

7. Conservation of Energy

- First law of Thermodynamics
  - Total Energy ***E***
- Different Terms of work and heat

$$E = m\left(\hat{u} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)$$

$$\left(\frac{dE}{dt}\right)_{MV} = \dot{Q} - \dot{W}$$

$$\left(\frac{dE}{dt}\right)_{MV} = \dot{Q}_V + \dot{Q}_S - \dot{W}_b - \dot{W}_S$$

7. Conservation of Energy

- Heat Rates

$$\dot{Q}_V = \int_V \dot{q}_V dV \quad \dot{Q}_S = -\int_S \dot{q}_s \cdot \mathbf{n}\, dS = -\int_V \nabla \cdot \dot{q}_s dV$$

- Work Rates

$$\dot{W}_b = -\int_V (\mathbf{f}_b \cdot \mathbf{v}) dV \qquad \dot{W}_S = -\int_S (\mathbf{f}_S \cdot \mathbf{v}) dS$$

$$\dot{W}_S = -\int_S [\Sigma \cdot \mathbf{v}] \cdot \mathbf{n}\, dS = -\int_V \nabla \cdot [\Sigma \cdot \mathbf{v}] dV = -\int_V \nabla \cdot [(-p\mathbf{I} + \boldsymbol{\tau}) \cdot \mathbf{v}] dV$$

$$\dot{W}_S = -\int_V (-\nabla \cdot [p\mathbf{v}] + \nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}]) dV$$

7. Conservation of Energy

- Applying RTT

$$B = E \Rightarrow b = \frac{dE}{dm} = \hat{u} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} = e$$

$$\left(\frac{dE}{dt}\right)_{MV} = \int_V \left[\frac{\partial}{\partial t}(\rho e) + \nabla\cdot[\rho\mathbf{v}e]\right]dV$$

$$= -\int_V \nabla\cdot\dot{q}_s dV + \int_V (-\nabla\cdot[p\mathbf{v}] + \nabla\cdot[\boldsymbol{\tau}\cdot\mathbf{v}])dV + \int_V (\mathbf{f}_b\cdot\mathbf{v})dV + \int_V \dot{q}_V dV$$

7. Conservation of Energy

- Collecting Terms to one side

$$\int_V \left[\frac{\partial}{\partial t}(\rho e) + \nabla \cdot [\rho \mathbf{v} e] + \nabla \cdot \dot{q}_s + \nabla \cdot [p\mathbf{v}] - \nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}] - \mathbf{f}_b \cdot \mathbf{v} - \dot{q}_V\right] dV = 0$$

- For any CV :

$$\frac{\partial}{\partial t}(\rho e) + \nabla \cdot [\rho \mathbf{v} e] = -\nabla \cdot \dot{q}_s - \nabla \cdot [p\mathbf{v}] + \nabla \cdot [\boldsymbol{\tau} \cdot \mathbf{v}] + \mathbf{f}_b \cdot \mathbf{v} + \dot{q}_V$$

7. Conservation of Energy – in terms of specific Internal Energy

- From the conservation of momentum eq. we have

$$\frac{\partial}{\partial t}[\rho \mathbf{v}] + \nabla \cdot \{\rho \mathbf{v}\mathbf{v}\} = \mathbf{f}$$

- Its dot product in velocity vector would result in

$$\left[\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot \{\rho \mathbf{v}\mathbf{v}\}\right] \cdot \mathbf{v} = \mathbf{f} \cdot \mathbf{v}$$

- After some manipulations:

$$\frac{\partial}{\partial t}(\rho \mathbf{v} \cdot \mathbf{v}) - \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \nabla \cdot [\rho(\mathbf{v} \cdot \mathbf{v})\mathbf{v}] - \rho \mathbf{v} \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}] = \mathbf{f} \cdot \mathbf{v}$$

- Rearranging and Collecting Terms

$$\frac{\partial}{\partial t}(\rho \mathbf{v} \cdot \mathbf{v}) + \nabla \cdot [\rho(\mathbf{v} \cdot \mathbf{v})\mathbf{v}] - \mathbf{v} \cdot \underbrace{\rho\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right]}_{= \mathbf{f}} = \mathbf{f} \cdot \mathbf{v}$$

7. Conservation of Energy – in terms of specific Internal Energy

- Using the general form of the conservation of linear momentum, we would get

$$\frac{\partial}{\partial t}\left(\rho\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)+\nabla\cdot\left[\rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\right]=-\mathbf{v}\cdot\nabla p+\mathbf{v}\cdot[\nabla\cdot\boldsymbol{\tau}]+\mathbf{f}_b\cdot\mathbf{v}$$

- It can be rewritten as

$$\frac{\partial}{\partial t}\left(\rho\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)+\nabla\cdot\left[\rho\left(\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right)\mathbf{v}\right]$$
$$=-\nabla\cdot[p\mathbf{v}]+p\nabla\cdot\mathbf{v}+\nabla\cdot[\boldsymbol{\tau}\cdot\mathbf{v}]-(\boldsymbol{\tau}:\nabla\mathbf{v})+\mathbf{f}_b\cdot\mathbf{v}$$

- Thus, using the definition of specific internal energy

$$\frac{\partial}{\partial t}(\rho\hat{u})+\nabla\cdot[\rho\mathbf{v}\hat{u}]=-\nabla\cdot\dot{q}_s-p\nabla\cdot\mathbf{v}+(\boldsymbol{\tau}:\nabla\mathbf{v})+\dot{q}_V$$

7. Conservation of Energy – in terms of specific Enthalpy

- Its definition

$$\hat{u} = \hat{h} - \frac{p}{\rho}$$

- Thus, after some algebraic manipulations:

$$\frac{\partial}{\partial t}(\rho \hat{h}) + \nabla \cdot [\rho \mathbf{v} \hat{\mathrm{h}}] = -\nabla \cdot \dot{q}_s + \frac{Dp}{Dt} + (\boldsymbol{\tau} : \nabla \mathbf{v}) + \dot{q}_V$$

7. Conservation of Energy – in terms of specific total Enthalpy

- Its definition

$$e = \hat{u} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} = \hat{h} - \frac{p}{\rho} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} = \hat{h}_0 - \frac{p}{\rho}$$

- Thus, after some algebraic manipulations:

$$\frac{\partial}{\partial t}(\rho\hat{h}_0) + \nabla\cdot[\rho\mathbf{v}\hat{h}_0] = -\nabla\cdot\dot{q}_s + \frac{\partial p}{\partial t} + \nabla\cdot[\boldsymbol{\tau}\cdot\mathbf{v}] + \mathbf{f}_b\cdot\mathbf{v} + \dot{q}_V$$

7. Conservation of Energy – in terms of Temperature

- Based on thermodynamic relations

$$d\hat{h} = c_p dT + \left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p\right] dp$$

- The substantial derivative of specific enthalpy is

$$\frac{\partial}{\partial t}(\rho\hat{h}) + \nabla \cdot [\rho\mathbf{v}\hat{h}] = \rho\frac{D\hat{h}}{Dt} = \rho c_p \frac{DT}{Dt} + \rho\left[\hat{V} - T\left(\frac{\partial \hat{V}}{\partial T}\right)_p\right]\frac{DP}{Dt}$$

$$= \rho c_p \frac{DT}{Dt} + \rho\left[\frac{1}{\rho} - T\left(\frac{\partial(1/\rho)}{\partial T}\right)_p\right]\frac{DP}{Dt}$$

$$= \rho c_p \frac{DT}{Dt} + \left[1 + \left(\frac{\partial(Ln\rho)}{\partial(LnT)}\right)_p\right]\frac{DP}{Dt}$$

7. Conservation of Energy – in terms of Temperature

- Using the eq. for conservation of energy in terms of specific enthalpy, we would get

$$\rho c_p \frac{DT}{Dt} = -\nabla \cdot \dot{q}_s - \left(\frac{\partial(Ln\rho)}{\partial(LnT)}\right)_p \frac{Dp}{Dt} + (\boldsymbol{\tau} : \nabla \mathbf{v}) + \dot{q}_V$$

- By expanding the substantial derivative

$$c_p \left[\frac{\partial}{\partial t}(\rho T) + \nabla \cdot [\rho \mathbf{v} T]\right] = -\nabla \cdot \dot{q}_s - \left(\frac{\partial(Ln\rho)}{\partial(LnT)}\right)_p \frac{Dp}{Dt} + (\boldsymbol{\tau} : \nabla \mathbf{v}) + \dot{q}_V$$

7. Conservation of Energy – in terms of Temperature

- The Heat Flux term, $\dot{q}_S$

$$\dot{q}_s = -[k\nabla T]$$

- Thus,

$$c_p\left[\frac{\partial}{\partial t}(\rho T) + \nabla\cdot[\rho\mathbf{v}T]\right] = \nabla\cdot[k\nabla T] - \left(\frac{\partial(Ln\rho)}{\partial(LnT)}\right)_p\frac{Dp}{Dt} + (\boldsymbol{\tau}:\nabla\mathbf{v}) + \dot{q}_V$$

7. Conservation of Energy – in terms of Temperature

- The expression for $(\boldsymbol{\tau} : \nabla \mathbf{v})$, in Cartesian 3D is given by

$$(\tau : \nabla \mathbf{v}) = \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2 + \mu \left(\begin{array}{l} 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 \\ + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \end{array} \right)$$

- We can define Ψ and Φ as

$$\Psi = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2$$

$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2$$

7. Conservation of Energy – in terms of Temperature

- Thus, we get:

$$c_p\left[\frac{\partial}{\partial t}(\rho T)+\nabla\cdot[\rho\mathbf{v}T]\right]=\nabla\cdot[k\nabla T]-\left(\frac{\partial(Ln\,\rho)}{\partial(Ln\,T)}\right)_p\frac{Dp}{Dt}+\lambda\Psi+\mu\Phi+\dot{q}_V$$

- For later reference,

$$\frac{\partial}{\partial t}(\rho c_p T)+\nabla\cdot[\rho c_p\mathbf{v}T]=\nabla\cdot[k\nabla T]\underbrace{+\rho T\frac{Dc_p}{Dt}-\left(\frac{\partial(Ln\,\rho)}{\partial(Ln\,T)}\right)_p\frac{Dp}{Dt}+\lambda\Psi+\mu\Phi+\dot{q}_V}_{Q^T}$$

$$\frac{\partial}{\partial t}(\rho c_p T)+\nabla\cdot[\rho c_p\mathbf{v}T]=\nabla\cdot[k\nabla T]+Q^T$$

7. Conservation of Energy – in terms of Temperature

$$\frac{\partial}{\partial t}(\rho c_p T) + \nabla \cdot [\rho c_p \mathbf{v} T] = \nabla \cdot [k \nabla T] + Q^T$$

- Special Cases,

  - The Dissipation term Φ , has negligible values except for large velocity gradients at supersonic speeds
  - For incompressible fluids, the continuity eq. implies that $\Psi = 0$ thus, $(\partial(Ln\rho)/\partial(LnT)) = 0.$

    and for this case,

$$\frac{\partial}{\partial t}(\rho c_p T) + \nabla \cdot [\rho c_p \mathbf{v} T] = \nabla \cdot [k \nabla T] + \underbrace{\dot{q}_V + \rho T \frac{Dc_p}{Dt}}_{Q^T}$$

7. Conservation of Energy – in terms of Temperature

$$\frac{\partial}{\partial t}(\rho c_p T) + \nabla \cdot [\rho c_p \mathbf{v} T] = \nabla \cdot [k \nabla T] + Q^T$$

- Special Cases,
  - For solids, density is constant, the velocity is zero, k would be considered constant too, thus

$$\rho c_p \frac{\partial T}{\partial t} = k \nabla^2 T + \dot{q}_V$$

  - For the ideal gases, $(\partial(Ln\rho)/\partial(LnT)) = -1$ thus the eq. reduces to

$$c_p \left[\frac{\partial}{\partial t}(\rho T) + \nabla \cdot [\rho \mathbf{v} T] \right] = \nabla \cdot [k \nabla T] + \frac{Dp}{Dt} + \lambda \Psi + \mu \Phi + \dot{q}_V$$

8. Non-dimensionalization Process

- Governing eqs. In 3D Cartesian System

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{\partial}{\partial t}(\rho u)+\frac{\partial}{\partial x}(\rho uu)+\frac{\partial}{\partial y}(\rho vu)+\frac{\partial}{\partial z}(\rho wu)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho vv)+\frac{\partial}{\partial z}(\rho wv)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\rho g$$

$$\frac{\partial}{\partial t}(\rho w)+\frac{\partial}{\partial x}(\rho uw)+\frac{\partial}{\partial y}(\rho vw)+\frac{\partial}{\partial z}(\rho ww)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

$$c_p\left[\frac{\partial}{\partial t}(\rho T)+\frac{\partial}{\partial x}(\rho uT)+\frac{\partial}{\partial y}(\rho vT)+\frac{\partial}{\partial z}(\rho wT)\right]=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)$$

8. Non-dimensionalization Process

- Boussinesq Approximation

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho vv)+\frac{\partial}{\partial z}(\rho wv)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\rho g$$

- Density using Taylor Expansion

$$\rho=\rho|_{T=T_\infty}+\left.\frac{d\rho}{dT}\right|_{T=T_\infty}(T-T_\infty)$$

- Volume Expansion coeff.

$$\beta=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$$

- Thus, the density becomes

$$\rho=\rho_\infty[1-\beta(T-T_\infty)]$$

- And Finally

$$\frac{\partial}{\partial t}(\rho v)+\frac{\partial}{\partial x}(\rho uv)+\frac{\partial}{\partial y}(\rho vv)+\frac{\partial}{\partial z}(\rho wv)$$
$$=-\frac{\partial}{\partial y}(p+\rho gy)+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g\beta(T-T_\infty)$$

8. Non-dimensionalization Process

- Non-dimensional form of variables

$$\hat{x} = \frac{x}{L}, \hat{y} = \frac{y}{L}, \hat{z} = \frac{z}{L}$$

$$\hat{u} = \frac{u}{\mu/(\rho L)}, \hat{v} = \frac{v}{\mu/(\rho L)}, \hat{w} = \frac{w}{\mu/(\rho L)}$$

$$\hat{t} = \frac{t}{\rho L^2/\mu}$$

$$\hat{p} = \frac{p + \rho g y}{\mu^2/(\rho L^2)}$$

$$\hat{T} = \frac{T - T_\infty}{T_{\max} - T_\infty}$$

8. Non-dimensionalization Process

- Non-dimensionalization of some of the terms in governing eqs.

$$\frac{\partial u}{\partial x} = \frac{\partial[\mu\hat{u}/(\rho L)]}{\partial(L\hat{x})} = \frac{\mu/(\rho L)}{L}\frac{\partial\hat{u}}{\partial\hat{x}} = \frac{\mu}{\rho L^2}\frac{\partial\hat{u}}{\partial\hat{x}}$$

$$\frac{\partial}{\partial t}(\rho u) = \frac{\partial(\mu\hat{u}/L)}{\partial(\rho L^2\hat{t}/\mu)} = \frac{\mu/L}{\rho L^2/\mu}\frac{\partial\hat{u}}{\partial\hat{t}} = \frac{\mu^2}{\rho L^3}\frac{\partial\hat{u}}{\partial\hat{t}}$$

$$\frac{\partial}{\partial t}(\rho uu) = \frac{\partial[\mu^2/(\rho L^2)\hat{u}\hat{u}]}{\partial(L\hat{x})} = \frac{\mu^2/(\rho L^2)}{L}\frac{\partial}{\partial\hat{x}}(\hat{u}\hat{u}) = \frac{\mu^2}{\rho L^3}\frac{\partial}{\partial\hat{x}}(\hat{u}\hat{u})$$

$$\hat{p} = \frac{p+\rho g y}{\mu^2/(\rho L^2)} \Rightarrow \frac{\partial\hat{p}}{\partial\hat{x}} = \frac{\partial\{(p+\rho g y)/[\mu^2/(\rho L^2)]\}}{\partial(x/L)}$$

$$= \frac{\rho L^3}{\mu^2}\frac{\partial p}{\partial x} \Rightarrow \frac{\partial p}{\partial x} = \frac{\mu^2}{\rho L^3}\frac{\partial\hat{p}}{\partial\hat{x}}$$

8. Non-dimensionalization Process

- Non-dimensionalization of some of the terms in governing eqs.

$$\mu \frac{\partial^2 u}{\partial x^2} = \mu \frac{\partial^2 [\mu \hat{u}/(\rho L)]}{\partial (L\hat{x})^2} = \mu \frac{\mu/(\rho L)}{L^2} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} = \frac{\mu^2}{\rho L^3} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}$$

$$\rho g \beta (T - T_\infty) = \rho g \beta (T_{\max} - T_\infty) \hat{T} = \rho g \beta (\Delta T) \hat{T}$$

$$\frac{\partial}{\partial t}(\rho T) = \frac{\partial \left[\rho \left(T_\infty + \Delta T \hat{T}\right)\right]}{\partial (\rho L^2 \hat{t}/\mu)} = \frac{\mu \Delta T}{L^2} \frac{\partial \hat{T}}{\partial \hat{t}}$$

$$\frac{\partial}{\partial x}(\rho u T) = \frac{\partial \left(\mu \hat{u} \left(T_\infty + \Delta T \hat{T}\right)/L\right)}{\partial (L\hat{x})} = \frac{\mu T_\infty}{L^2} \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\mu \Delta T}{L^2} \frac{\partial}{\partial \hat{x}} \left(\hat{u} \hat{T}\right)$$

$$k \frac{\partial^2 T}{\partial x^2} = k \frac{\partial^2 \left(T_\infty + \Delta T \hat{T}\right)}{\partial (L\hat{x})^2} = \frac{k \Delta T}{L^2} \frac{\partial^2 \hat{T}}{\partial \hat{x}^2}$$

8. Non-dimensionalization Process

- Non-dimensionalized Governing Eqs.

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} = 0$$

$$\frac{\partial \hat{u}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{u}\hat{u}) + \frac{\partial}{\partial \hat{y}}(\hat{v}\hat{u}) + \frac{\partial}{\partial \hat{z}}(\hat{w}\hat{u}) = -\frac{\partial \hat{p}}{\partial \hat{x}} + \left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{z}^2}\right)$$

$$\frac{\partial \hat{v}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{u}\hat{v}) + \frac{\partial}{\partial \hat{y}}(\hat{v}\hat{v}) + \frac{\partial}{\partial \hat{z}}(\hat{w}\hat{v}) = -\frac{\partial \hat{p}}{\partial \hat{y}} + \left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{z}^2}\right) + Gr\hat{T}$$

$$\frac{\partial \hat{w}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{u}\hat{w}) + \frac{\partial}{\partial \hat{y}}(\hat{v}\hat{w}) + \frac{\partial}{\partial \hat{z}}(\hat{w}\hat{w}) = -\frac{\partial \hat{p}}{\partial \hat{z}} + \left(\frac{\partial^2 \hat{w}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{z}^2}\right)$$

$$\frac{\partial \hat{T}}{\partial \hat{t}} + \frac{\partial}{\partial \hat{x}}(\hat{u}\hat{T}) + \frac{\partial}{\partial \hat{y}}(\hat{v}\hat{T}) + \frac{\partial}{\partial \hat{z}}(\hat{w}\hat{T}) = \frac{1}{Pr}\left(\frac{\partial^2 \hat{T}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{T}}{\partial \hat{y}^2} + \frac{\partial^2 \hat{T}}{\partial \hat{z}^2}\right)$$

$$Gr = \frac{g\beta\Delta T L^3}{\nu^2}$$

$$\nu = \frac{\mu}{\rho}$$

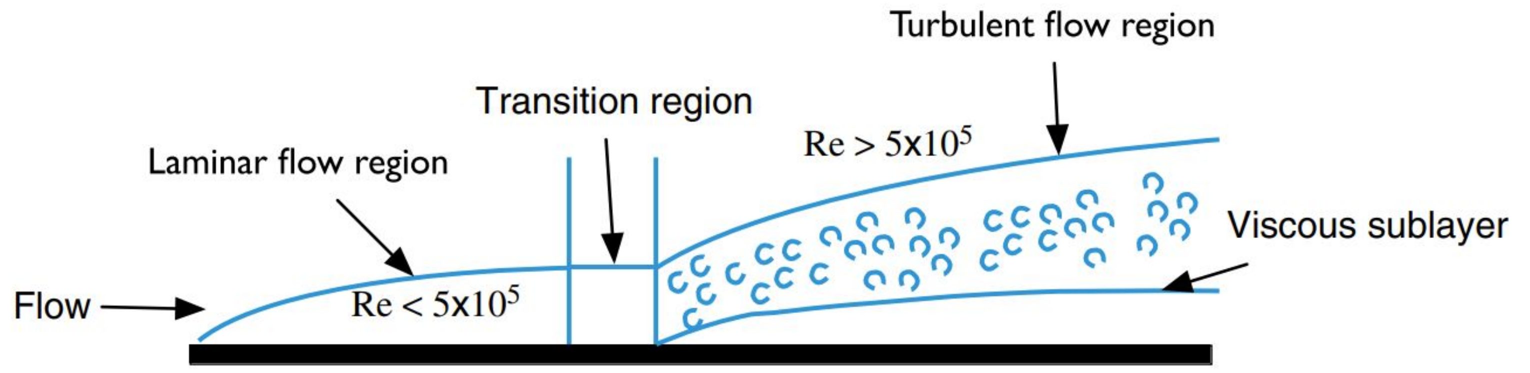
$$Pr = \frac{\mu c_p}{k}$$

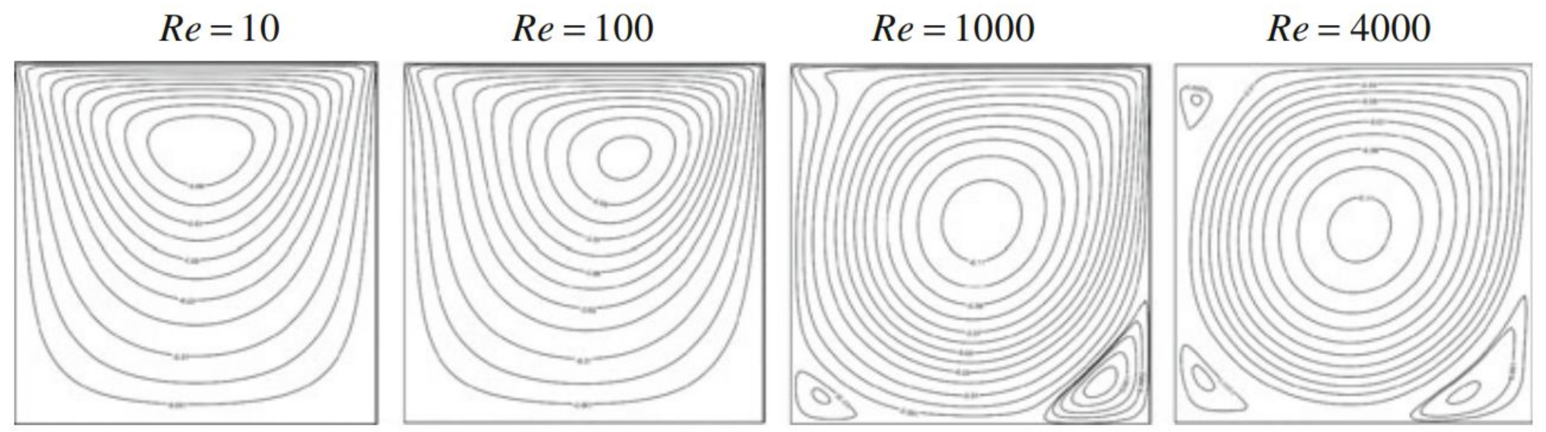
9. Dimensionless Numbers - Re

$$Re = \frac{\rho U L}{\mu}$$

- Definition

- A measure of relative importance of advection (inertia) to diffusion (viscous) momentum fluxes



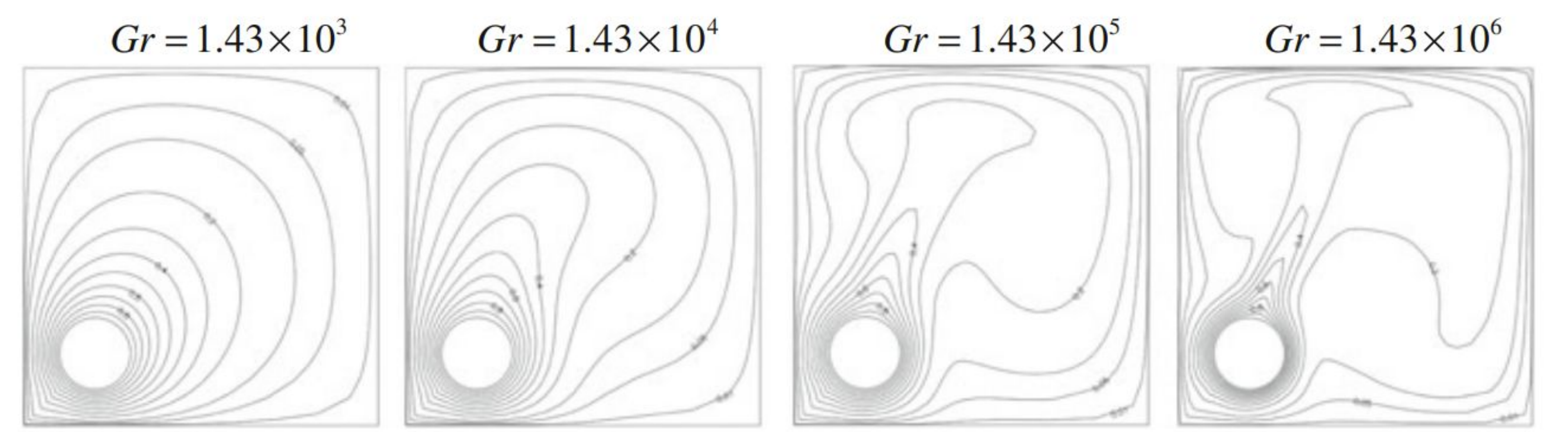


9. Dimensionless Numbers - Gr

- Definition

$$Gr = \frac{g\beta\Delta T L^3}{\nu^2}$$

- Represents the ratio of buoyant to viscous forces
- It plays the role of Re in Natural Convection

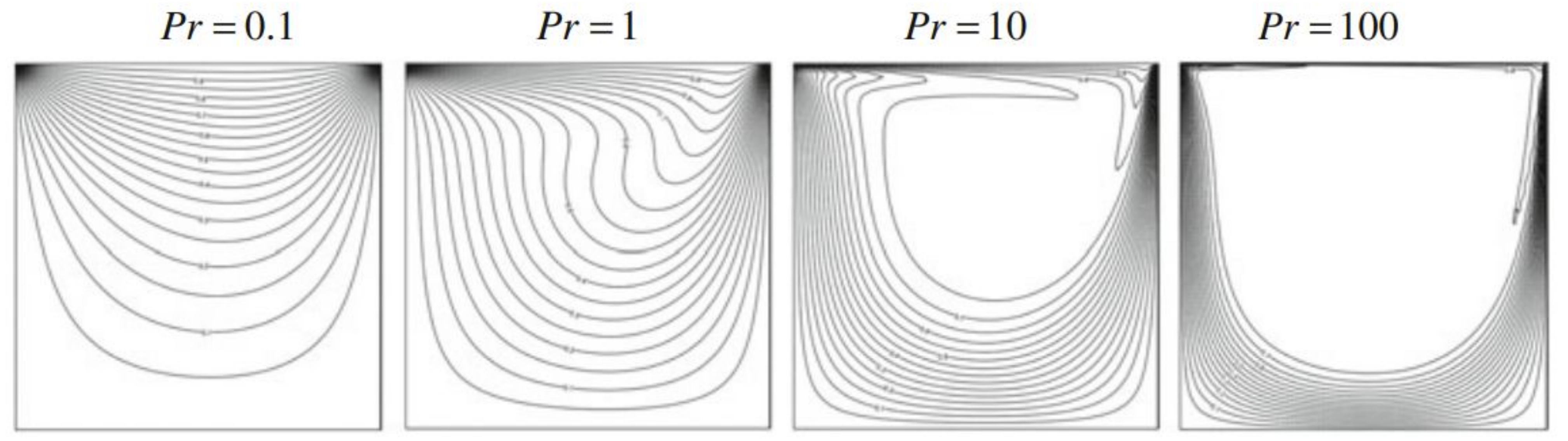


9. Dimensionless Numbers - Pr

- Definition

$$Pr = \frac{\mu c_p}{k} = \frac{\mu/\rho}{k/\rho c_p} = \frac{\nu}{\alpha}$$

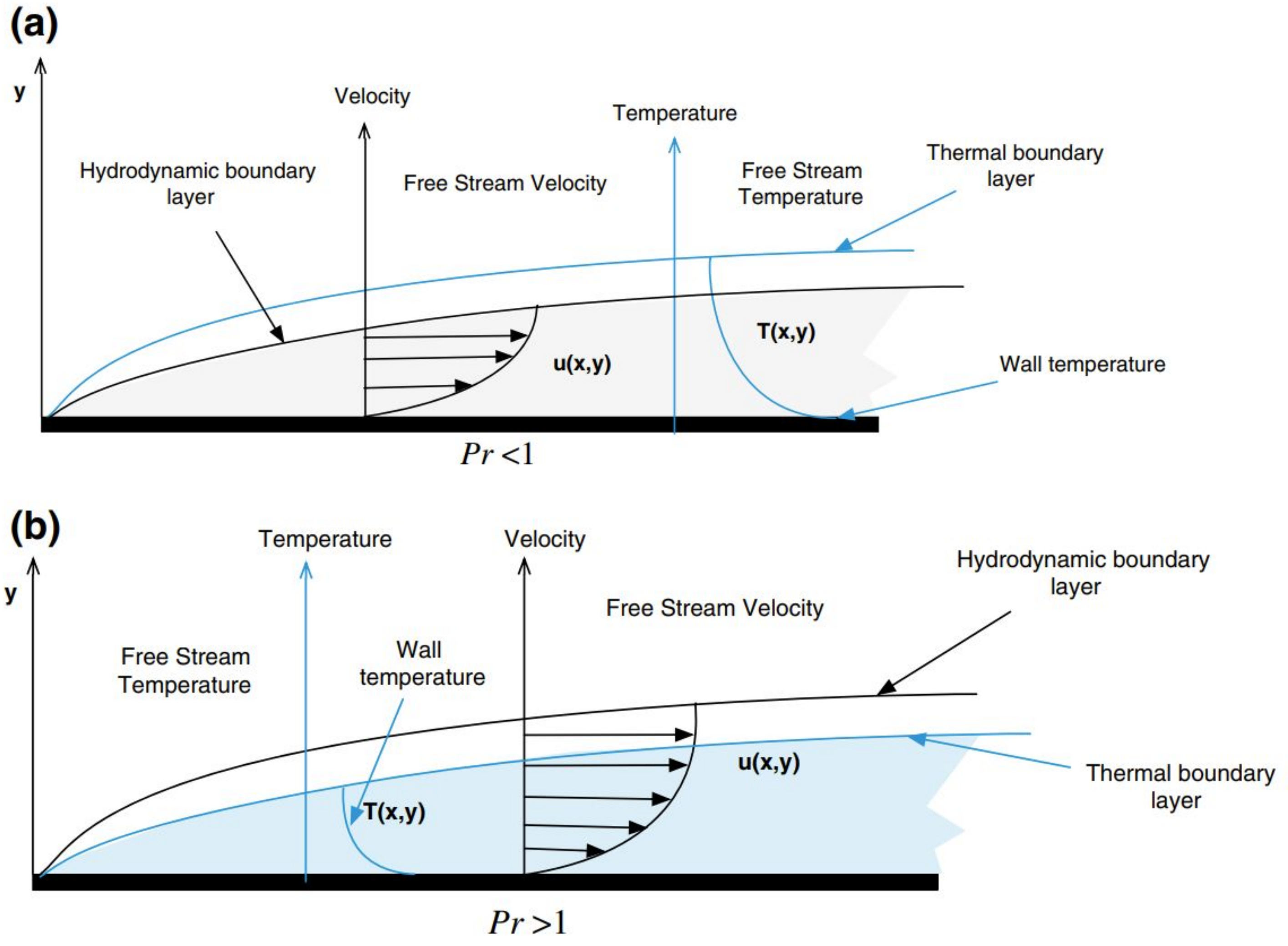
- Represents the ratio of momentum diffusivity to thermal diffusivity
- Also represents the ratio of hydrodynamic boundary layer to thermal boundary layer



9. Dimensionless Numbers - Pr

- Definition

$$Pr = \frac{\mu c_p}{k} = \frac{\mu/\rho}{k/\rho c_p} = \frac{\nu}{\alpha}$$



9. Dimensionless Numbers - Pe

- Definition

$$Pe = \frac{\rho U L c_p}{k} = \frac{UL}{\alpha} = Re^* Pr$$

$$Pe = \frac{UL}{D} = Re^* Sc$$

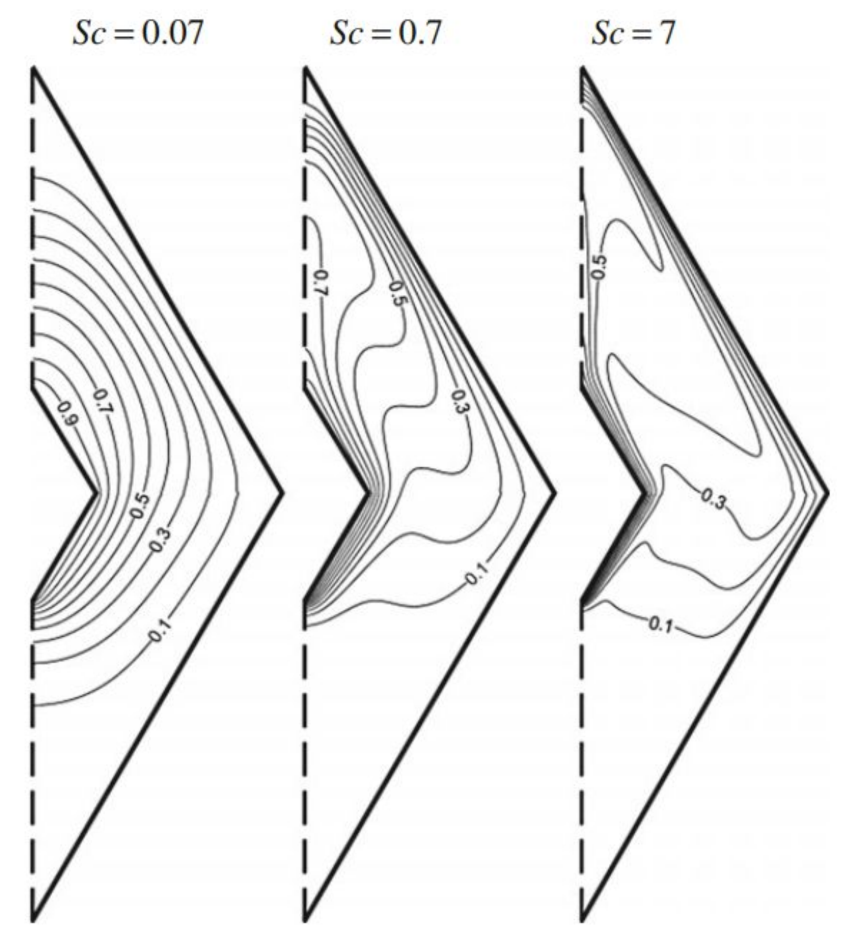
- Ratio of the advective transport rate of a physical quantity to its diffusive transport rate

9. Dimensionless Numbers - Sc

$$Sc = \frac{\nu}{D}$$

- Definition

- Like Pr but for mass transfer
- Represents the ratio of the momentum diffusivity to mass diffusivity
- Also relates the thickness of hydrodynamic boundary layer to mass transfer boundary layer



9. Dimensionless Numbers - Nu

- Definition

$$Nu = \frac{hL}{k}$$

- Not brought up in non-dimensionalization of conservation equations but widely used in information of convective transport

9. Dimensionless Numbers - Mach

- Definition

$$M = \frac{|\mathbf{v}|}{a}$$

- Ratio of speed of an object moving through a fluid and the local speed of sound
- General relation for local speed of sound

$$a = \sqrt{\gamma \left(\frac{\partial p}{\partial \rho}\right)_T}$$

- For an ideal gas, it reduces to

$$a = \sqrt{\gamma R T}$$

- If M<0.2 the flow can be treated as incompressible
- Suubsonic Sonic Supersonic Hypersonic

9. Dimensionless Numbers - Ec

- Definition

$$Ec = \frac{\mathbf{v} \cdot \mathbf{v}}{c_p \Delta T}$$

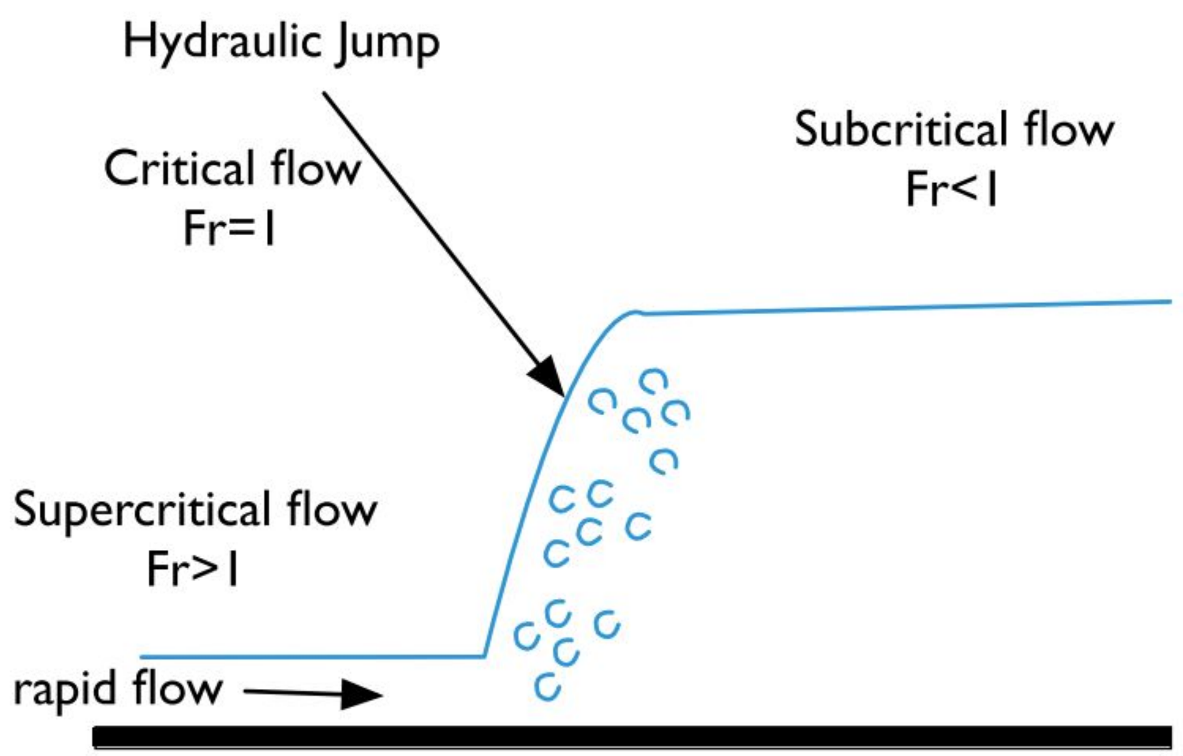
- Relates the kinetic energy of the flow to its enthalpy
- It appears as a factor multiplying the viscous dissipation

9. Dimensionless Numbers - Fr

- Definition

$$Fr = \frac{U}{\sqrt{gL}}$$

- A measure of the resistance of partially immersed objects moving through fluids



9. Dimensionless Numbers - We

- Definition

$$We = \frac{\rho U^2 L}{\sigma}$$

- Represents the ratio of inertia to surface tension forces
- It is helpful in analyzing multiphase flow involving interfaces between two different fluids, with curved surfaces such as droplets and bubbles

10. Exercises for Introductory Course

- Problem 01

Let $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ be three vectors given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 10 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 8 \\ -5 \\ -2 \end{bmatrix}$$

Find:

a. $\mathbf{v}_1 + \mathbf{v}_2, \quad \mathbf{v}_1 + 2\mathbf{v}_2, \quad 3\mathbf{v}_2 - 4\mathbf{v}_3$

b. $|\mathbf{v}_1|, |\mathbf{v}_2|, |\mathbf{v}_3|$

c. $\mathbf{v}_1 \cdot \mathbf{v}_2, \quad \mathbf{v}_3 \times \mathbf{v}_2, \quad \mathbf{v}_2 \cdot (\mathbf{v}_1 \times \mathbf{v}_3)$

d. A unit vector in the direction of $(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$

10. Exercises for Introductory Course

- Problem 02

Let **i**, **j** and **k** be unit vectors in the x, y, and z direction, respectively, and let **v** be any vector, which in a Cartesian coordinate system is given by

$$\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

Prove that

$$\mathbf{v} = C[\mathbf{i} \times (\mathbf{v} \times \mathbf{i}) + \mathbf{j} \times (\mathbf{v} \times \mathbf{j}) + \mathbf{k} \times (\mathbf{v} \times \mathbf{k})]$$

where C is a constant to be determined.

10. Exercises for Introductory Course

- Problem 03

Find ∇s if s is the scalar function given by

a. $s = y^2 e^{2x-3z}$

b. $s = \mathrm{Ln}(x + y^2 + z^3)$

c. $s = \tan^{-1}\left(\frac{x}{yz}\right)$

10. Exercises for Introductory Course

- Problem 04

Find the Laplacian of the scalar s $(\nabla^2 s)$ for the cases when s is given by:

a. $s = x^3 + z^2 e^{2y-3x}$
b. $s = z + \mathrm{Ln}(x + y)$
c. $s = \sin^{-1}(x + y + z)$

10. Exercises for Introductory Course

- Problem 05

Use the divergence theorem to evaluate the integral $\iint_{\partial F} (6x\mathbf{i} + 4y\mathbf{j}) \cdot d\mathbf{F}$ where the surface is a sphere defined as $\partial F \rightarrow x^2 + y^2 + z^2 = 10$.

10. Exercises for Introductory Course

- Problem 06

Show that for an incompressible flow of constant viscosity the following holds:

$$\nabla \cdot \left\{ \mu \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{T}} \right] \right\} = \mu \nabla^2 \mathbf{v}$$

10. Exercises for Introductory Course

- Problem 07

A steady incompressible flow field is defined by the following velocity vector:

$$\mathbf{v} = (x + y)\mathbf{i} + (y + z)\mathbf{j} + 2(x - z)\mathbf{k}$$

(a) Verify that it satisfies the continuity equation.
(b) Assuming constant viscosity μ, calculate the viscous stress tensor $\boldsymbol{\tau}$.
(c) Denoting the fluid density by ρ and neglecting body forces, develop an equation for the pressure gradient.

10. Exercises for Introductory Course

- Problem 08

Starting from the incompressible version of the Navier-Stokes equations derive simplified equations based on the following assumptions:

(a) Viscous effects are much more significant than any effects of fluid acceleration, i.e.,

$$\frac{\partial}{\partial t}(\mathbf{v}) + \nabla \cdot [\mathbf{vv}] \ll \nabla \cdot [\mu \nabla \mathbf{v}]$$

which corresponds to $Re = \rho UL/\mu \ll 1$ (Stokes Equations).

(b) Inertial effects dominate and viscous effects are considered to be negligible throughout the flow domain, i.e.,

$$\frac{\partial}{\partial t}(\mathbf{v}) + \nabla \cdot [\mathbf{vv}] \gg \nabla \cdot [\mu \nabla \mathbf{v}]$$

which corresponds to $Re = \rho UL/\mu \gg 1$ (Euler equations).

(c) Derive the Bernoulli equation from momentum conservation with the following hypothesis: one dimensional steady state conditions of a frictionless fluid $\mu = 0$.

Thanks for your
time and attention